Overview

1. Illustrate a fundamental choice: Benefits vs. Costs
2. Develop formal approach and notation
3. Introduce basic decision criterion: Prospective Benefit, $B(t)$, and Prospective Cost, $C(t)$

Example

Assume a 1-year magazine subscription costs $20 and a 2-year subscription costs $35. As a special offer, these prices are guaranteed for the next four years if you subscribe today.

Which would you prefer?

<table>
<thead>
<tr>
<th></th>
<th>1-Yr</th>
<th>2-Yr</th>
<th>Diff. (2Yr-1Yr)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0</td>
<td>-20</td>
<td>-15</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>-20</td>
<td>+20</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-20</td>
<td>-15</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-20</td>
<td>+20</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fundamental choice: Benefits, Costs

<table>
<thead>
<tr>
<th></th>
<th>Series</th>
<th>Series</th>
<th>Diff. (2-1) = Series - Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0</td>
<td>-15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>+20</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-15</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>+20</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Assume $i = 0.03$ per year

- $F_B = 20(1.03)^2 + 20 = $41.22
- $F_C = 15(1.03)^3 + 15(1.03) = $31.84

Since $F_B > F_C$, choose 2-Yrs

A more formal approach

An economic decision criterion includes both:
1. Measure of Worth
2. Decision Rule
A special interest rate: MARR

\[ i = \text{minimum attractive rate of return (MARR)} \]
also known as --
- marginal growth rate,
- discount rate,
- cutoff rate,
- hurdle rate,
- yield,
- among others

The role of the MARR

<table>
<thead>
<tr>
<th>( t )</th>
<th>( A_{jt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100</td>
</tr>
<tr>
<td>1</td>
<td>105</td>
</tr>
</tbody>
</table>

Interest Rate = 5%

An economical investment?

If MARR < 5%; Yes
If MARR > 5%; No

Values of the MARR

Typically, the MARR for:
1) individuals represents the min. attractive opportunities to invest in money markets
2) corporations represents the min. attractive opportunities to invest in the company

Some notation

\[ A_{jt} = \text{net cash flow, where:} \]
\[ j = \text{index on opportunities} \]
\[ A_{jt} > 0 \text{ is a net receipt,} \]
\[ A_{jt} < 0 \text{ is a net expense, and} \]
\[ A_{jt} = 0 \text{ for} \ t < 0 \text{ and} \ t > N \]
\[ A_{jt} = B_{jt} - C_{jt} \text{ where:} \]
\[ B_{jt} = A_{jt} \text{ if} \ A_{jt} > 0, \text{ else} B_{jt} = 0, \]
\[ C_{jt} = -A_{jt} \text{ if} \ A_{jt} < 0, \text{ else} C_{jt} = 0 \]

Basic Criterion: Benefits and Costs

Measure of Worth

\[ B_j(i) = \sum_{t=0}^{N} B_{jt}(1+i)^{-t}, \text{ where} \]
\[ i = \text{MARR} \]

\[ C_j(i) = \sum_{t=0}^{N} C_{jt}(1+i)^{-t} \]

Typically, \( T = 0 \) or \( N \)

Decision Rule

Accept (prefer) \( j \) if \( B_j(i) > C_j(i) \), otherwise reject (not prefer) \( j \)

An example

A manufacturer is considering buying 10 robots to spray paint its product on the assembly line. Each robot costs $200,000 and has an expected life of 9 years. The cost to install all the robots is $45,000. Each robot is expected to reduce labor costs by $50,000 a year but will increase energy costs by $15,000 a year.

If the MARR = 10% per year, are the robots economical?
Fundamentals of an Economic Decision

... an example

<table>
<thead>
<tr>
<th>$t$</th>
<th>Problem Data</th>
<th>$A_{RI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10(-200K)-45K</td>
<td>-2,045K</td>
</tr>
<tr>
<td>1</td>
<td>10(+50K-15K)</td>
<td>350K</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>350K</td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$N$</td>
<td>9</td>
<td>350K</td>
</tr>
</tbody>
</table>

Fundamentals of an Economic Decision

... an example

<table>
<thead>
<tr>
<th>$t$</th>
<th>$A_{RI}$ = $B_{RI}$ - $C_{RI}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 - 2,045K</td>
</tr>
<tr>
<td>1</td>
<td>350K</td>
</tr>
<tr>
<td>2</td>
<td>350K</td>
</tr>
<tr>
<td>...</td>
<td>$N$=9</td>
</tr>
</tbody>
</table>

$B_{R}(.1) = 350(F/A,.1,9) = \$4752.8K$

$C_{R}(.1) = 2045(F/P,.1,9) = \$4822.0K$

Reject; note only 1.4% difference!

Fundamentals of an Economic Decision

Summary

◆ A fundamental choice: Benefits vs. Costs
◆ Formal approach and notation
  ➔ Criterion = MOW + DR
  ➔ MARR
  ➔ $A_{j0}, B_{j0}, C_{j0}$
◆ Basic decision criterion:
  ➔ $B_{j}(i), C_{j}(i)$

Future Worth, Present Worth, Annual Worth

Overview

Three classic economic criteria based on worth:
◆ Future Worth, $FW_{j}(i)$
◆ Present Worth, $PW_{j}(i)$
◆ Annual Worth, $AW_{j}(i)$

Future Worth

Measure of Worth

$FW_{j}(i) = B_{j}(i) - C_{j}(i)$, where $i = MARR$

$FW_{j}(i) = \sum_{t=0}^{N} B_{j}(1+i)^{T-t} - \sum_{t=0}^{N} C_{j}(1+i)^{T-t}$

$FW_{j}(i) = \sum_{t=0}^{N} [B_{j} - C_{j}](1+i)^{T-t}$

$FW_{j}(i) = \sum_{t=0}^{N} A_{j}(1+i)^{T-t}$, Typically $T = N$

Decision Rule

Accept $j$ if $FW_{j}(i) > 0$, else reject
A manufacturer is considering buying 10 robots to spray paint its product on the assembly line. Each robot costs $200,000 and has an expected life of 9 years. The cost to install all the robots is $45,000. Each robot is expected to reduce labor costs by $50,000 a year but will increase energy costs by $15,000 a year. If the MARR = 10% per year, are the robots economical?

\[ t \quad A_{R_t} = B_{R_t} - C_{R_t} \]

<table>
<thead>
<tr>
<th></th>
<th>Route 1</th>
<th>Route 2</th>
<th>Route 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-75K</td>
<td>-125K</td>
<td>-100K</td>
</tr>
<tr>
<td>1-14</td>
<td>23K</td>
<td>28K</td>
<td>25.5K</td>
</tr>
<tr>
<td>15</td>
<td>68K</td>
<td>53K</td>
<td>60.5K</td>
</tr>
</tbody>
</table>

\[ PW_{R1}(0.08) = -75K + 23K(P/A,0.08,14) + 68K(P/F,0.08,15) = +136.1K \]

\[ PW_{R2}(0.08) = +122.6K \]

\[ PW_{R3}(0.08) = +129.3K \]

Choose R1?: \( PW_{R1} > PW_{R2}; PW_{R3} \)?
An equivalent decision rule

For any two alternatives \( j = A_1, A_2 \):
\[ PWA_{A1-A2}(i) = PWA_{A1}(i) - PWA_{A2}(i) \]

Thus, equivalent decision rule is:
prefer \( A_1 \) over \( A_2 \) when
\[ PWA_{A1}(i) > PWA_{A2}(i) \]
and no capital rationing!
(Same is true for Future Worth)

Example, revisited (again)

\[
\begin{array}{cccc}
 t & A_{R1,t} & A_{R2,t} & A_{R3,t} \\
 0 & 75K & 125K & 100K \\
 1-14 & 23K & 28K & 25.5K \\
 15 & 68K & 53K & 60.5K \\
\end{array}
\]

\[
\begin{align*}
AW_{R1}(.08) &= +15.9K > 0 \\
AW_{R2}(.08) &= +14.3K > 0 \\
AW_{R3}(.08) &= +15.1K > 0 \\
AW_{R2-R1}(.08) &= -1.6K < 0 \\
AW_{R3-R1}(.08) &= -0.8K < 0
\end{align*}
\]

\( AW \) also has equivalent decision rule

Summary

Three classic economic criteria based on worth:
- Future Worth, \( FW_j(i) \)
- Present Worth, \( PW_j(i) \)
- Annual Worth, \( AW_j(i) \)

Internal Rate of Return

Overview

- Definition
- Illustrative examples
- Potential problems unique to this criterion
Measure of Worth (most common definition)

\[ i^* = \{ i \mid \text{PW}_j(i) = 0 \} \]

or

\[ i^* = \{ i \mid \text{FW}_j(i) = 0 \} \]

Decision Rules

Accept investment \( j \)
if \( i^* > \text{MARR} \), else reject

Accept borrowing \( j \)
if \( i^* < \text{MARR} \), else reject

**Example: An investment**

Assume you can purchase a photocopy machine for $20,000. Its estimated life and salvage value is 5 years and $5,000, respectively. It will save $7,000 a year in outside vendor copying, but it will cost $1,000 a year in maintenance, repair, paper, ink, etc.

Assume the MARR is 9% per year. What is the IRR?

**Example: The net cash flows**

\[
\begin{array}{cccc}
 t & A_{ct}  \\
 0 & -20,000  \\
 1 & 6,000  \\
 2 & 6,000  \\
 3 & 6,000  \\
 4 & 6,000  \\
 5 & 11,000  \\
\end{array}
\]

\[
\begin{align*}
\text{PW}_C(0.15) &= 2598.81  \\
\text{PW}_C(0.25) &= -2225.92  \\
\text{PW}_C(0.199021) &= +0.00089211  \\
\end{align*}
\]

\[ i_C^* \approx 0.20 > 0.09 \]

**Example: A borrowing**

Assume you can finance the copier with a loan from the supplier. For a down payment of $5,000, you can finance the remainder. The loan is to be repaid in 5 years with annual payments of $3,561.00.

What is the interest rate (IRR) of the loan?

**Example: The net cash flows**

\[
\begin{array}{cccc}
 t & A_{ct}  \\
 0 & +15,000  \\
 1 & -3,561  \\
 2 & -3,561  \\
 3 & -3,561  \\
 4 & -3,561  \\
 5 & -3,561  \\
\end{array}
\]

\[
\begin{align*}
\text{PW}_L(i) &= 15,000 - 3561(P/A,i,5)  \\
i_L^* \approx 0.06 < 0.09  \\
\end{align*}
\]

**Investment? Borrowing?**

\[
\begin{array}{cccc}
 t & 1-\text{Yr} & 2-\text{Yr} & A_{2Yr-1Yr} & A_{1Yr-2Yr}  \\
 0 & -20 & -35 & -15 & +15  \\
 1 & -20 & +20 & -20  \\
 2 & -20 & -35 & -15 & +15  \\
 3 & -20 & +20 & -20  \\
 4 &  &  &  \\
\end{array}
\]

\[
\begin{align*}
\text{PW}_{2:1}(i^*) &= -15+20(1+i)^{-1} -15(1+i)^{-2} +20(1+i)^{-3} = 0  \\
i_{2:1}^* & = 0.335 > \text{MARR} = 0.03; \text{ Accept? Reject?}  \\
\text{PW}_{1:2}(i^*) &= +15-20(1+i)^{-1} +15(1+i)^{-2} -20(1+i)^{-3} = 0  \\
i_{1:2}^* & = 0.335 > \text{MARR} = 0.03; \text{ Accept? Reject?}  \\
\end{align*}
\]
**Internal Rate of Return**

**What is an “investment?”**

<table>
<thead>
<tr>
<th>Time</th>
<th>Capital Invested</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-20,000</td>
</tr>
<tr>
<td>1</td>
<td>6,000</td>
</tr>
<tr>
<td>2</td>
<td>6,000</td>
</tr>
<tr>
<td>3</td>
<td>6,000</td>
</tr>
<tr>
<td>4</td>
<td>6,000</td>
</tr>
<tr>
<td>5</td>
<td>11,000</td>
</tr>
</tbody>
</table>

**Capital remains invested at all times**

**What is a “borrowing?”**

<table>
<thead>
<tr>
<th>Time</th>
<th>Capital Borrowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>+15,000</td>
</tr>
<tr>
<td>1</td>
<td>-3,561</td>
</tr>
<tr>
<td>2</td>
<td>-3,561</td>
</tr>
<tr>
<td>3</td>
<td>-3,561</td>
</tr>
<tr>
<td>4</td>
<td>-3,561</td>
</tr>
<tr>
<td>5</td>
<td>-3,561</td>
</tr>
</tbody>
</table>

**Capital remains borrowed at all times**

**Accept or reject if MARR = 0.15?**

<table>
<thead>
<tr>
<th>Time</th>
<th>Capital Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-5,000</td>
</tr>
<tr>
<td>1</td>
<td>3,000</td>
</tr>
<tr>
<td>2</td>
<td>-1,400</td>
</tr>
<tr>
<td>3</td>
<td>-1,000</td>
</tr>
<tr>
<td>4</td>
<td>8,400</td>
</tr>
</tbody>
</table>

\[ j^* = 0.20 > 0.15, \text{ accept} \]

**Accept or reject if MARR = 0.15?**

<table>
<thead>
<tr>
<th>Time</th>
<th>Capital Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2,000</td>
</tr>
<tr>
<td>1</td>
<td>-1,200</td>
</tr>
<tr>
<td>2</td>
<td>900</td>
</tr>
<tr>
<td>3</td>
<td>800</td>
</tr>
<tr>
<td>4</td>
<td>-3,300</td>
</tr>
</tbody>
</table>

\[ j^* = 0.10 < 0.15, \text{ accept} \]

**Yet another example . . .**

It is common in the oil industry to “acidize” a well to increase production. Acids are pumped into the well to cleanse the porous rock near the bottom of the well to increase oil production.

Assume \( j = 0 \) represents “do nothing,” and \( j = 1 \) represents acidizing. MARR = 20%.

\[ i_{1-0}^* = 1.35 > 0.20; \text{ and} \]
\[ i_{1-0}^* = 0.0 < 0.20 \]
### Internal Rate of Return

**One final example... MARR = 0.15**

<table>
<thead>
<tr>
<th>t</th>
<th>A_{A1,t}</th>
<th>A_{A2,t}</th>
<th>A_{A2-A1,t}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10,000</td>
<td>-20,000</td>
<td>-10,000</td>
</tr>
<tr>
<td>1</td>
<td>5,500</td>
<td>-5,500</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>5,500</td>
<td>-5,500</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>5,500</td>
<td>40,000</td>
<td>+34,500</td>
</tr>
</tbody>
</table>

\[ i^*_j = 0.30 \quad 0.26 \quad 0.24 \]

\[ PW_j(0.15) = 2,558 \quad 6,300 \quad +3,742 > 0 \]

**Note:** \( i^*_j A_{A1-A2} \neq i^*_j A_{A1} - i^*_j A_{A2} \); IRR does not have an equivalent decision rule!

### Summary

- **Definition**
- Two decision rules
- **Illustrative examples**
  - Investment
  - Borrowing
- **Mixed Investment/Borrowing**
- **No equivalent decision rule**

### Benefit/Cost Ratio, and Payback Period

**Overview**

- **Benefit/Cost Ratio**
- **Payback Period**
  - (a secondary criterion)

**Notes:**
- The presentation on Benefit/Cost Ratio is based on the “strict” definition of benefits and costs. See text for the “flexible” definition.
- The definition of Payback Period is a conceptual one, and some situations can require a more detailed definition.

### Benefit/Cost Ratio

**Measure of Worth**

\[ B_j(i) = \sum_{t=0}^{N} B_j(1+i)^{-t} \]

\[ BCR_j(i) = \frac{B_j(i)}{C_j(i)} = \sum_{t=0}^{N} C_j(1+i)^{-t} \]

where \( i = MARR \) & \( T = 0 \) or \( N \)

**Decision Rule**

Accept (prefer) \( j \) if \( BCR_j(i) > 1.0 \), otherwise reject (not prefer) \( j \).

### An example... MARR = 0.10

<table>
<thead>
<tr>
<th>t</th>
<th>A_{A1,t}</th>
<th>A_{A2,t}</th>
<th>A_{A2-A1,t}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2,000</td>
<td>-3,000</td>
<td>-1,000</td>
</tr>
<tr>
<td>1</td>
<td>900</td>
<td>1,200</td>
<td>300</td>
</tr>
<tr>
<td>2</td>
<td>900</td>
<td>1,200</td>
<td>300</td>
</tr>
<tr>
<td>3</td>
<td>900</td>
<td>1,200</td>
<td>300</td>
</tr>
<tr>
<td>4</td>
<td>900</td>
<td>1,200</td>
<td>300</td>
</tr>
<tr>
<td>5</td>
<td>900</td>
<td>1,200</td>
<td>300</td>
</tr>
</tbody>
</table>

\[ BCR_{(0.10)} = 1.71 \quad 1.52 \quad 1.137 > 1 \]

\[ PW_{(0.10)} = 1,412 \quad 1,549 \quad +137 > 0 \]

**Note:** BCR has no equivalent decision rule!
Payback Period (Secondary Criterion)

Measure of “Worth” (actually, liquidity)

\[ N_j^* = \{ N \mid \sum_{t=0}^{N} A_{jt}(1+i)^{-t} = 0 \} \]

Decision Rule
Accept investment \( j \) if \( N_j^* < N_{\text{max}} \), else reject (for multiple alternatives, prefer the shortest payback period).

Benefit/Cost Ratio, and Payback Period

<table>
<thead>
<tr>
<th>( t )</th>
<th>( A_{A_1,t} )</th>
<th>( A_{A_2,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-9,760</td>
<td>-10,000</td>
</tr>
<tr>
<td>1</td>
<td>5,000</td>
<td>4,600</td>
</tr>
<tr>
<td>2</td>
<td>5,000</td>
<td>4,600</td>
</tr>
<tr>
<td>3</td>
<td>5,000</td>
<td>4,600</td>
</tr>
<tr>
<td>4</td>
<td>5,000</td>
<td>4,600</td>
</tr>
<tr>
<td>5</td>
<td>5,000</td>
<td>4,600</td>
</tr>
</tbody>
</table>

\( i_{A_1^*} = 0.43 \)
\( \text{PW}_{A_1}(0.25) = 3,686 \)
\( N_{A_1^*} = 3 \)

Benefit/Cost Ratio, and Payback Period

<table>
<thead>
<tr>
<th>( t )</th>
<th>( A_{A_2,t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-10,000</td>
</tr>
<tr>
<td>1</td>
<td>-1994</td>
</tr>
<tr>
<td>2</td>
<td>-5275</td>
</tr>
<tr>
<td>3</td>
<td>-7900</td>
</tr>
<tr>
<td>4</td>
<td>-7900(1.25) = -6594</td>
</tr>
<tr>
<td>5</td>
<td>-7900(1.25) = -6594</td>
</tr>
<tr>
<td></td>
<td>-10000(1.25) = -12500</td>
</tr>
</tbody>
</table>

\( 3 < N_{A_2^*} \leq 4 \)

Summary

- Fundamental concepts of a decision
  - Benefit, \( B_j(i) \), & Cost, \( C_j(i) \)
- Five classic criteria:
  - Future Worth, \( FW_j(i) \)
  - Present Worth, \( PW_j(i) \)
  - Annual Worth, \( AW_j(i) \)
  - Internal Rate of Return, \( i_j^* \)
- Popular secondary criterion:
  - Payback Period, \( N_j^* \)

Home Mortgage Refinancing - A Case Study

- Homeowner’s situation
- Selecting a criterion
- Existing loan
- Refinancing option
- Defining a new alternative
- Analysis using break-even internal rate of return
Home Mortgage Refinancing - A Case Study

Homeowner's situation

- GT faculty member recently bought a home with a 30-year mortgage
- Refinancing opportunity available at lower interest rate, 15-year period
- Question: Refinance?
  - Lower interest rate
  - Transaction (closing) costs
  - Higher monthly payment because of shorter term

Selecting a criterion

- FW, PW, AW: what MARR?
- B/C ratio: classification of B?
- IRR: long periods, 180 and 360
  Applied to 60-month period
- Another criterion:
  Future cash amount
  New alternative defined for use with this criterion

Existing loan (Option 1)

- Initial balance, $P = $202,000
- $N = (30)(12) = 360$ months
- $i = 9%/12 = 0.75\%$ per month
- Monthly payment for principal and interest ($P&I)$:
  $A = P(A/P, 0.75\%, 360)$
  $= 202,000(0.00804623)$
  $= $1,625.34

New loan (Option 2)

The new loan would be for the same amount as the existing loan, but at a lower interest rate of 0.625\% per month, and for a period of 15 years. Also, there would be a $2,000 transaction fee (closing cost).

New loan (Option 2)

$P = $202,000

$A = $1,872.56/month
$A$ includes $P&I$
**How to compare?**

Very few people live in a house long enough to pay off a 30-year mortgage.

One way to compare the two alternatives is to calculate the remaining loan balances after 5, 10, and 15 years.

---

**Existing loan (1), after 5 yrs.**

\( P = 202,000 \)

\( i = 0.75\% \text{ per month} \)

\( A = 1,625.34/\text{month} \)

\( N = 300 \text{ months} \)

- After 5 years of payments, the remaining loan balance is:
  \( P = A(P/A, 0.75\%, 300) = 1,625.34(119.1616) = 193,678 \)

- After 10 years: $180,648
- After 15 years: $160,248
- Slow decline is typical of 30-year mortgages

**New loan (2), after 5 yrs.**

\( P = 202,000 \)

\( i = 0.625\% \text{ per month} \)

\( A = 1,872.56/\text{month} \)

\( N = 120 \text{ months} \)

- After 5 years of payments, the remaining loan balance is:
  \( P = A(P/A, 0.625\%, 120) = 1,872.56(84.2447) = 157,753 \)

- After 10 years: $93,451
- After 15 years: **zero**!
  Loan is completely paid off.

---

**Remaining loan balance (1)**

- After 5 years of payments, the remaining loan balance is:
  \( P = A(P/A, 0.75\%, 300) = 1,625.34(119.1616) = 193,678 \)

- After 10 years: $180,648
- After 15 years: $160,248
- Slow decline is typical of 30-year mortgages

**Remaining loan balance (2)**

- After 5 years of payments, the remaining loan balance is:
  \( P = A(P/A, 0.625\%, 120) = 1,872.56(84.2447) = 157,753 \)

- After 10 years: $93,451
- After 15 years: **zero**!
  Loan is completely paid off.

**Apples and oranges (1 vs. 2)**

The difficulty with the previous comparisons is the difference in the cash flows paid to the mortgage company.

Solution? Construct a hypothetical alternative (Option 3) where the homeowner keeps the existing mortgage contract but replicates the cash flow of the new loan. The excess payments reduce the principal balance. Then, a comparison can be made on remaining loan balances.
Home Mortgage Refinancing - A Case Study

### Hypothetical alternative (3)

- \( P = \$202,000 \)
- \( i \) remains 0.75% per month
- \( A = \$1,625.34 + 247.22 \)
- \( = \$1,872.56/\text{month} \)

\( \$2,000 \)

\[ N = ? \]

### Effect on loan balance (3)

- After 5 years the \( \$2,000 \)
  would have the effect of reducing the loan balance by
  \[ F = P(F/P, 0.75\%, 60) \]
  \[ = \$2,000(1.56568) = \$3,131 \]
- The effect of the \( \$247.22 \)
  per month extra payment is
  \[ F = A(F/A, 0.75\%, 60) \]
  \[ = \$247.22(75.4241) = \$18,646 \]

### Compute loan balance (3)

- Effect of the \( \$2,000 \) after
  - 10 years: \( \$4,903 \)
  - 15 years: \( \$7,676 \)
- Effect of the \( \$247.22 \) after
  - 10 years: \( \$47,841 \)
  - 15 years: \( \$93,549 \)
- Next step: compute remaining loan balances for hypothetical alternative (Option 3)

### Rate-of-return analysis

Define Option 4:
the homeowner deposits the excess \( \$2,000 \) and \( \$247.22 \) per month into an account that grows in 5 years to \( \$35,925 \), which is the difference after 5 years between Options 1 and 2:
\[ \$193,678 - \$157,753. \]
Home Mortgage Refinancing - A Case Study

**Rate-of-return analysis**

\[ F = \$35,925 \]

\[ A = \$247.22/\text{mo.} \]

\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

\[ 60 \]

\[ \$2,000 \]

**Rate-of-return = 2.084\% per month**

---

Result: \[ i^* = 2.084\% / \text{month} \]

Effective annual rate is

\[ (1 + 0.0284)^{12} - 1 = 28\% \]

This is a rather high rate that might not be achieved with a commercial account. So the new loan is preferred.

---

**Summary**

- New criterion: future cash
  - Existing loan
  - New loan
  - Hypothetical alternative
- Rate-of-return analysis for 5-year period on difference between alternatives

---

**Multiple Alternatives - Part 1**

**Overview**

- Define relationships possible among investment alternatives
- Present typical examples
- Demonstrate selection process for investment alternatives
  - PW for independent investments
  - Rate-of-return for mutually exclusive investments

---

**Multiple Alternatives - Part 1**

**Definition**

**Independent alternatives:**

Investment alternatives where the selection of one does not restrict the selection of another.
Multiple Alternatives - Part 1

Example 1

A  Replace gas-fired boiler with a more fuel-efficient unit

B  Replace plant electric control system with “smart” system that reduces peak load from 450 to 350 kW

Definition

Mutually exclusive alternatives:

Investment alternatives where the selection of one precludes the selection of another.

Example 2

A  Replace gas-fired boiler with a more fuel-efficient unit from Manufacturer X

B  Replace gas-fired boiler with a more fuel-efficient unit from Manufacturer Y

Options are:

Φ  Do Nothing

A  Replace gas-fired boiler with unit from Mfg. X

B  Replace gas-fired boiler with unit from Mfg. Y

Consider: Φ, A, B

Definition

Contingent alternatives:

Investment alternatives where the selection of one depends on the selection of another.
Multiple Alternatives - Part 1

Example 3

A  Install stand-by 350 kW diesel generator for refrigeration system

B  Install “smart” electric control system that reduces peak load from 450 to 350 kW

A by itself doesn’t make much sense, so A depends on B.

Example 3

Options are \( \Phi, B, (A + B) \)

\( \Phi \)  Do Nothing

B  Install “smart” electric control system that reduces peak load from 450 to 350 kW (benefit from rate reduction based on peak use)

\( (A + B) \)  Install generator plus smart control system (benefit from rate reduction based on peak use plus rate reduction based on interruptible supply)

Example 4

A  Company Z gains the right to distribute its products in zone A

B  Company Z gains the right to distribute its products in zone B

(A + B)  Company Z gains the right to distribute its products in zones A and B

We would expect some savings in transport costs when serving both zones.

Definition

Complementary alternatives:

Two or more alternatives are complementary when the relative worth of their combination is more than the sum of the individual relative worths.
Multiple Alternatives - Part 1

Example 4

Consider these options:

Φ  Do Nothing
A  Distribute products in zone A
B  Distribute products in zone B
C  Distribute products in both zones A and B, with cost reduction reflecting combined transport operation

Example 5

A  Replace gas-fired boiler for heating system with a more fuel-efficient unit
B  Install system of sensors and closers at doors, windows, and other openings to reduce heat loss during winter season
A + B technically feasible, but the savings are not additive
Consider: Φ, A, B, and C based on A + B reflecting loss of savings

Example 6

Consider:

Φ, A, B, (A + B)

PW_A(15%) = 0  by definition
PW_A(15%) = −300,000
+ 115,000(P/A, 15%, 4)
= −300,000 + 115,000(2.85498)
= $28,323 > 0, so A is acceptable

PW_B(15%) = −400,000
+ 125,000(P/A, 15%, 5)
= −400,000 + 125,000(3.35216)
= $19,019 > 0, so B is acceptable

PW_A+B(15%) = 28,323 + 19,019
= $47,342 > 0, so (A + B) is acceptable

Result: Select (A + B)

(Note: if mutually exclusive, select A)

Reminder: To compute PW of a combination of two or more independent investment alternatives, simply add the PW values of the individual investments.)
Example 7

Mutually exclusive alternatives C and D:

<table>
<thead>
<tr>
<th>Alternative</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment</td>
<td>$200,000</td>
<td>$260,000</td>
</tr>
<tr>
<td>Project life</td>
<td>6 years</td>
<td>6 years</td>
</tr>
<tr>
<td>Annual net cash flow</td>
<td>$85,000</td>
<td>$125,000</td>
</tr>
<tr>
<td>MARR</td>
<td>15%</td>
<td></td>
</tr>
</tbody>
</table>

No budget constraint

Use Rate-of-return to make decisions

Which alternative to select?

Consider:

\[ PWC(i) = -200,000 + 85,000 \cdot \left( \frac{P}{A}, i, 6 \right) \]

\[ \left( \frac{P}{A}, i, 6 \right) = \frac{200,000}{85,000} = 2.35294 \]

so \( i^* = 35.7\% > 15\% \), so accept C for now

\[ PWD_C(i) = -(260,000 - 200,000) + (125,000 - 85,000) \cdot \left( \frac{P}{A}, i, 6 \right) \]

\[ \left( \frac{P}{A}, i, 6 \right) = \frac{60,000}{40,000} = 1.5 \]

so \( i^* = 63.1\% > 15\% \), so accept D

Final choice is D

(If independent, select both since \( i^*_D = 42.3\% \))

Summary

- Defined relationships possible among investment alternatives
  - Independent
  - Mutually exclusive
  - Contingent
  - Complementary
  - Competitive
- Example of independent alternatives, with PW criterion
- Example of mutually exclusive alternatives, with IRR criterion

Multiple Alternatives - Part 2

Overview

- Present method for defining combinations of independent and mutually exclusive investment alternatives
- Demonstrate selection process for more involved situations
  - Rate-of-return
  - B/C ratio

Procedure

Combination alternatives:

Construct logical combinations of alternatives that are technically feasible.

Eliminate those that exceed the budget constraint, if any.
Multiple Alternatives - Part 2

**Example 1**

A manufacturing company is trying to expand its market into some new areas: E, F, and G.

There are a number of potential distributors for serving each of the new areas:

- area E: distributor V, W, and X
- area F: distributor X, Y, and Z
- area G: distributor V, W, and Z

**Example 1**

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>E, ( \phi )</td>
<td>0</td>
<td>F, ( \phi )</td>
</tr>
<tr>
<td>E, V</td>
<td>60,000</td>
<td>F, X</td>
</tr>
<tr>
<td>E, W</td>
<td>80,000</td>
<td>F, Y</td>
</tr>
<tr>
<td>E, X</td>
<td>90,000</td>
<td>F, Z</td>
</tr>
</tbody>
</table>

**Example 1**

Constraint 1: There can be only one distributor in an area.

Constraint 2: No distributor can serve in two or more areas, to prevent stretching sales staff.

Constraint 3: There is a limit on investment of $150,000 at the inception of the expansion program.

**Example 1** (a)

<table>
<thead>
<tr>
<th>Combin. Inv.</th>
<th>OK?</th>
<th>Combin. Inv.</th>
<th>OK?</th>
</tr>
</thead>
<tbody>
<tr>
<td>E, ( \phi ), F, G, V</td>
<td>0</td>
<td>E, F, G, V</td>
<td>50</td>
</tr>
<tr>
<td>E, V</td>
<td>60</td>
<td>E, V, F, G, V</td>
<td>110</td>
</tr>
<tr>
<td>E, W</td>
<td>80</td>
<td>E, W, F, G, V</td>
<td>130</td>
</tr>
<tr>
<td>E, X</td>
<td>90</td>
<td>E, X, F, G, V</td>
<td>140</td>
</tr>
<tr>
<td>E, F, G, V</td>
<td>130</td>
<td>E, F, G, V</td>
<td>145</td>
</tr>
</tbody>
</table>

**Example 1** (b)

<table>
<thead>
<tr>
<th>Combin. Inv.</th>
<th>OK?</th>
<th>Combin. Inv.</th>
<th>OK?</th>
</tr>
</thead>
<tbody>
<tr>
<td>E, F, G, V</td>
<td>45</td>
<td>E, F, G, V</td>
<td>95</td>
</tr>
<tr>
<td>E, V, F, G, V</td>
<td>--</td>
<td>E, F, G, V</td>
<td>--</td>
</tr>
<tr>
<td>E, F, G, V</td>
<td>115</td>
<td>E, F, G, V</td>
<td>130</td>
</tr>
</tbody>
</table>
Example 1 (c)

<table>
<thead>
<tr>
<th>Combin.</th>
<th>Inv.</th>
<th>OK?</th>
<th>Combin.</th>
<th>Inv.</th>
<th>OK?</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₀,F₀,GW</td>
<td>65</td>
<td>Y</td>
<td>E₀,F,Y,GW</td>
<td>115</td>
<td>Y</td>
</tr>
<tr>
<td>E₁,F₀,GW</td>
<td>125</td>
<td>Y</td>
<td>E₁,F,Y,GW</td>
<td>175</td>
<td>N</td>
</tr>
<tr>
<td>E₂,F₀,GW</td>
<td>N</td>
<td>N</td>
<td>E₂,F,Y,GW</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>E₃,F₀,GW</td>
<td>N</td>
<td>N</td>
<td>E₃,F,Y,GW</td>
<td>205</td>
<td>N</td>
</tr>
<tr>
<td>E₀,F,X,GW</td>
<td>135</td>
<td>N</td>
<td>E₀,F,Z,GW</td>
<td>150</td>
<td>Y</td>
</tr>
<tr>
<td>E₁,F,X,GW</td>
<td>195</td>
<td>N</td>
<td>E₁,F,Z,GW</td>
<td>210</td>
<td>N</td>
</tr>
<tr>
<td>E₂,F,X,GW</td>
<td>N</td>
<td>N</td>
<td>E₂,F,Z,GW</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>E₃,F,X,GW</td>
<td>N</td>
<td>N</td>
<td>E₃,F,Z,GW</td>
<td>240</td>
<td>N</td>
</tr>
</tbody>
</table>

Example 1 (d)

<table>
<thead>
<tr>
<th>Combin.</th>
<th>Inv.</th>
<th>OK?</th>
<th>Combin.</th>
<th>Inv.</th>
<th>OK?</th>
</tr>
</thead>
<tbody>
<tr>
<td>E₀,F₀,GZ</td>
<td>75</td>
<td>Y</td>
<td>E₀,F,Y,GZ</td>
<td>125</td>
<td>Y</td>
</tr>
<tr>
<td>E₁,F₀,GZ</td>
<td>135</td>
<td>Y</td>
<td>E₁,F,Y,GZ</td>
<td>185</td>
<td>N</td>
</tr>
<tr>
<td>E₂,F₀,GZ</td>
<td>155</td>
<td>N</td>
<td>E₂,F,Y,GZ</td>
<td>205</td>
<td>N</td>
</tr>
<tr>
<td>E₃,F₀,GZ</td>
<td>165</td>
<td>N</td>
<td>E₃,F,Y,GZ</td>
<td>215</td>
<td>N</td>
</tr>
<tr>
<td>E₀,F,X,GZ</td>
<td>145</td>
<td>N</td>
<td>E₀,F,Z,GZ</td>
<td>--</td>
<td>N</td>
</tr>
<tr>
<td>E₁,F,X,GZ</td>
<td>205</td>
<td>N</td>
<td>E₁,F,Z,GZ</td>
<td>--</td>
<td>N</td>
</tr>
<tr>
<td>E₂,F,X,GZ</td>
<td>225</td>
<td>N</td>
<td>E₂,F,Z,GZ</td>
<td>--</td>
<td>N</td>
</tr>
<tr>
<td>E₃,F,X,GZ</td>
<td>--</td>
<td>N</td>
<td>E₃,F,Z,GZ</td>
<td>--</td>
<td>N</td>
</tr>
</tbody>
</table>

Summary of procedure:

Form all logical combinations.
Delete those that violate any technical constraints.
Delete those that violate any budget constraint.
Apply PW to select the best combination.

Consider 3 mutually exclusive options:
A, B, and C, with cash flows as shown in the table.
Apply Rate-of-return to select the best option.

MARR = 15%

Project
<table>
<thead>
<tr>
<th>t</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–30,000</td>
<td>–15,000</td>
<td>–20,000</td>
</tr>
<tr>
<td>1</td>
<td>11,175</td>
<td>6,159</td>
<td>7,546</td>
</tr>
<tr>
<td>2</td>
<td>11,175</td>
<td>6,159</td>
<td>7,546</td>
</tr>
<tr>
<td>3</td>
<td>11,175</td>
<td>6,159</td>
<td>7,546</td>
</tr>
<tr>
<td>4</td>
<td>11,175</td>
<td>6,159</td>
<td>7,546</td>
</tr>
<tr>
<td>5</td>
<td>11,175</td>
<td>6,159</td>
<td>7,546</td>
</tr>
</tbody>
</table>

Sequence for analysis is B, C, A

\[ PW_{A}(i) = -15,000 + 6,159(P/A, i, 5) = 0 \]
\[ (P/A, i, 5) = 15,000/6,159 = 2.43546 \]
\[ i* = 30\% > 15\%, \text{ so accept B for now} \]

\[ PW_{C}(i) = -(20,000-15,000) + (7,546-6,159)(P/A, i, 5) = 0 \]
\[ (P/A, i, 5) = 5,000/1,387 = 3.60490 \]
\[ i* = 12.0\% < 15\%, \text{ so reject C} \]
B remains the defender
**Example 2**

\[
P_{WA}(i) = -(30,000 - 15,000) + (11,175 - 6,159)(P/A, i, 5) = 0
\]

\[
(P/A, i, 5) = 15,000/5,016 = 2.99043
\]

\[i^* = 20.0\% > 15\%, \text{ so accept A}
\]

Final selection is A

---

**Example 3**

Consider 2 mutually exclusive options:
A, and B, with cash flows as shown in the table.

Apply Benefit/cost ratio to select the best option.
MARR = 15%
Sequence for analysis is A, B.

<table>
<thead>
<tr>
<th>Project</th>
<th>Outflows</th>
<th>Inflows</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>(A)</td>
<td>(B)</td>
</tr>
<tr>
<td>0</td>
<td>20,000</td>
<td>35,000</td>
</tr>
<tr>
<td>1</td>
<td>40,000</td>
<td>57,000</td>
</tr>
<tr>
<td>2</td>
<td>60,000</td>
<td>56,000</td>
</tr>
<tr>
<td>3</td>
<td>80,000</td>
<td>42,000</td>
</tr>
</tbody>
</table>

\(B_{ij}(i)\) and \(C_{ij}(i)\) are defined (slightly) differently here compared to their initial definition (in which either \(B_j\) or \(C_j\) were equal to zero). Both definitions are used in practice and both are correct.

Consider \(B_A(15\%) = 232,284\)
Now \(C_A(15\%) = 152,753\)
so \(BCR_A(15\%) = 232,284/152,753 = 1.52 > 1\), so accept A for now

Consider \(B_B-A(15\%) = 252,355 - 232,284 = 20,071\)
Consider \(C_B-A(15\%) = 154,525 - 152,753 = 1,772\)
so \(BCR_B-A(15\%) = 20,071/1,772 = 11.35 > 1\), so accept B, final choice is B

---

**Summary**

- Method for defining combinations of independent and mutually exclusive investment alternatives: enumeration, eliminate ineligible combinations
- Demonstrated selection process for more involved situations
  - Rate-of-return, incremental method
  - B/C ratio, incremental method

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