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Multi-objective COMPASS & GO-POLARS

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This research work is
collaborated with D-SIMLAB Pte. Ltd.

Outline

- Introduction
  - Aircraft spare part allocation problem
  - The COMPASS Algorithm & SARs
  - Concept of Pareto Optimality
- The MO-COMPASS Algorithm
- Numerical Examples & Testing Results
- Go-Polars

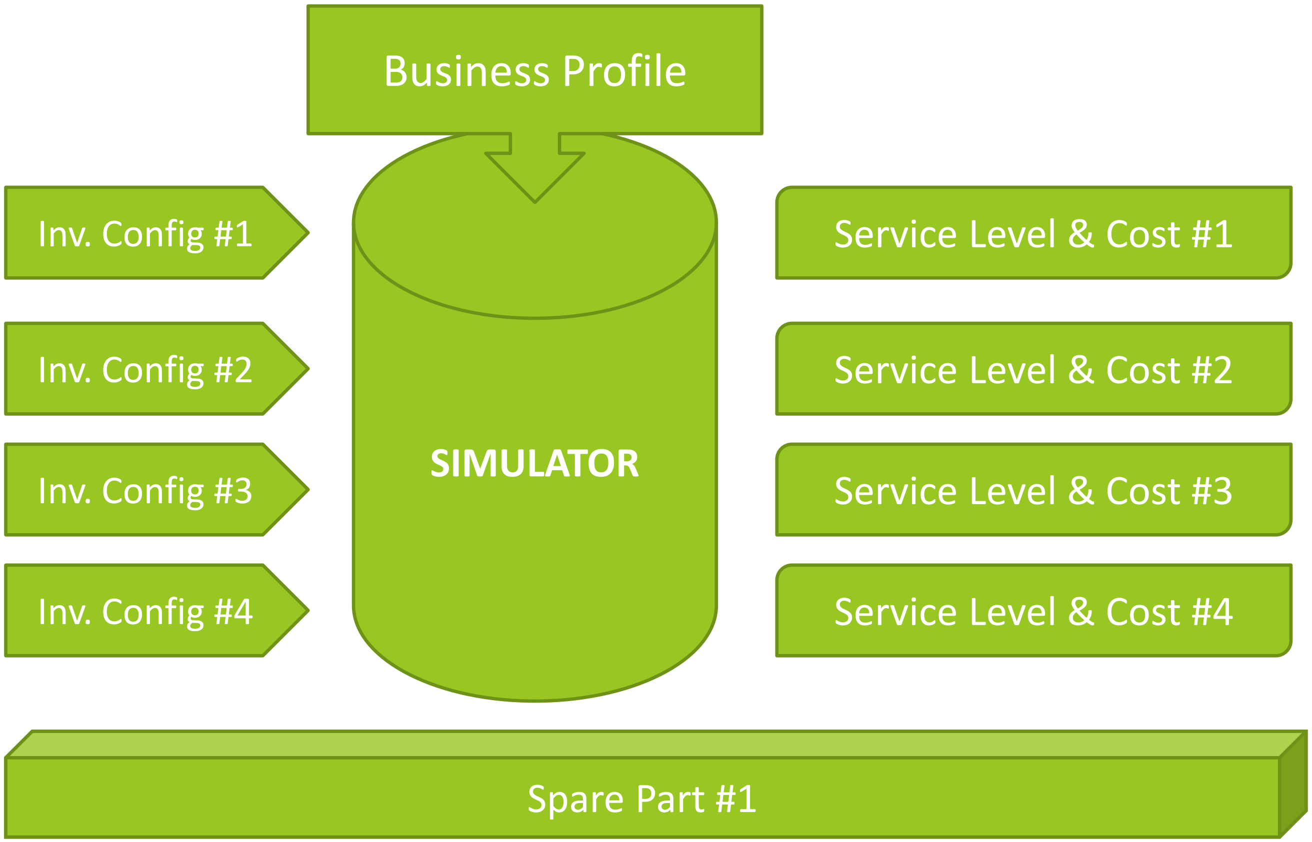
D-SIMSPAIR

- A flagship product of D-SIMLAB Technologies Pte. Ltd.
- A simulation based optimization solution for aerospace spare-part management

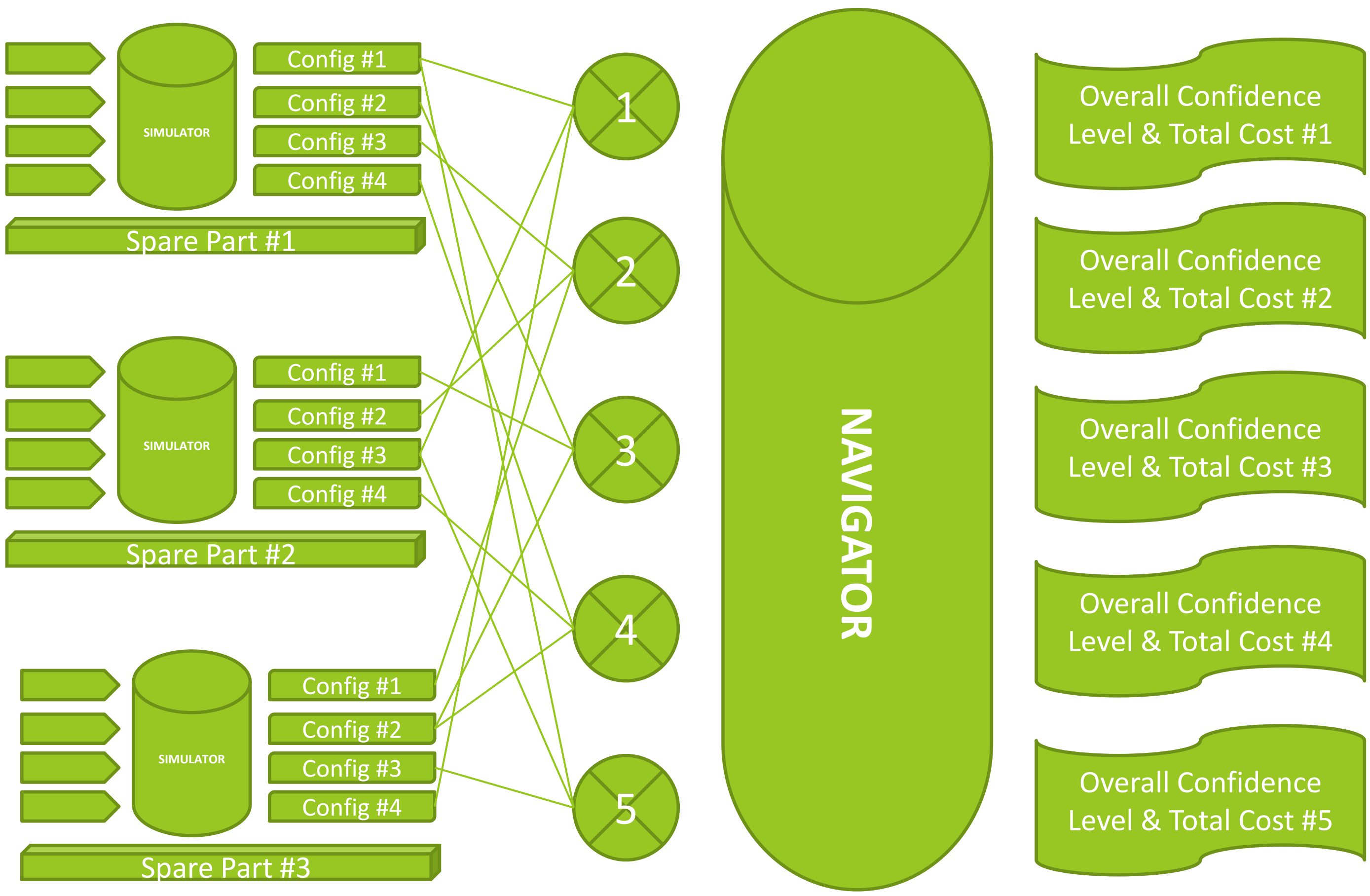
What is D-SIMSPAIR

- D-SIMSPAIR is to answer:
  - Where to store the spare parts?
    - Which Airport?
    - Which Warehouse?
  - How many parts to store in each airport?
  - What is the confidence level to meet committed service level?
  - How much is the associated cost?

PHASE - I ENUMERATION



PHASE - II NAVIGATION



Challenges

- Enumeration Process is too Long
  - Need to run simulation
  - Too many alternatives
- Navigation Process is not effective
  - GA does not take into account of the neighborhood structure

Opportunity

- Need an effective algorithm for the enumeration phase
- Multi-objective search algorithm to find the inventory level at each location for each component which
  - Minimize Cost
  - Maximize service level

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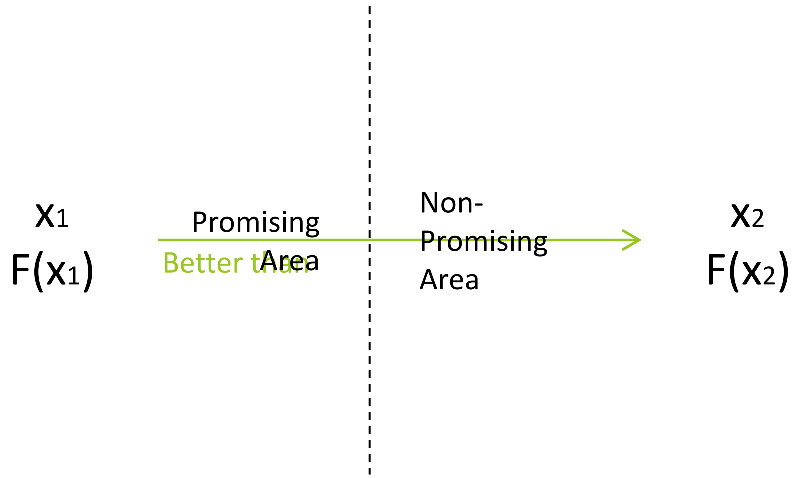
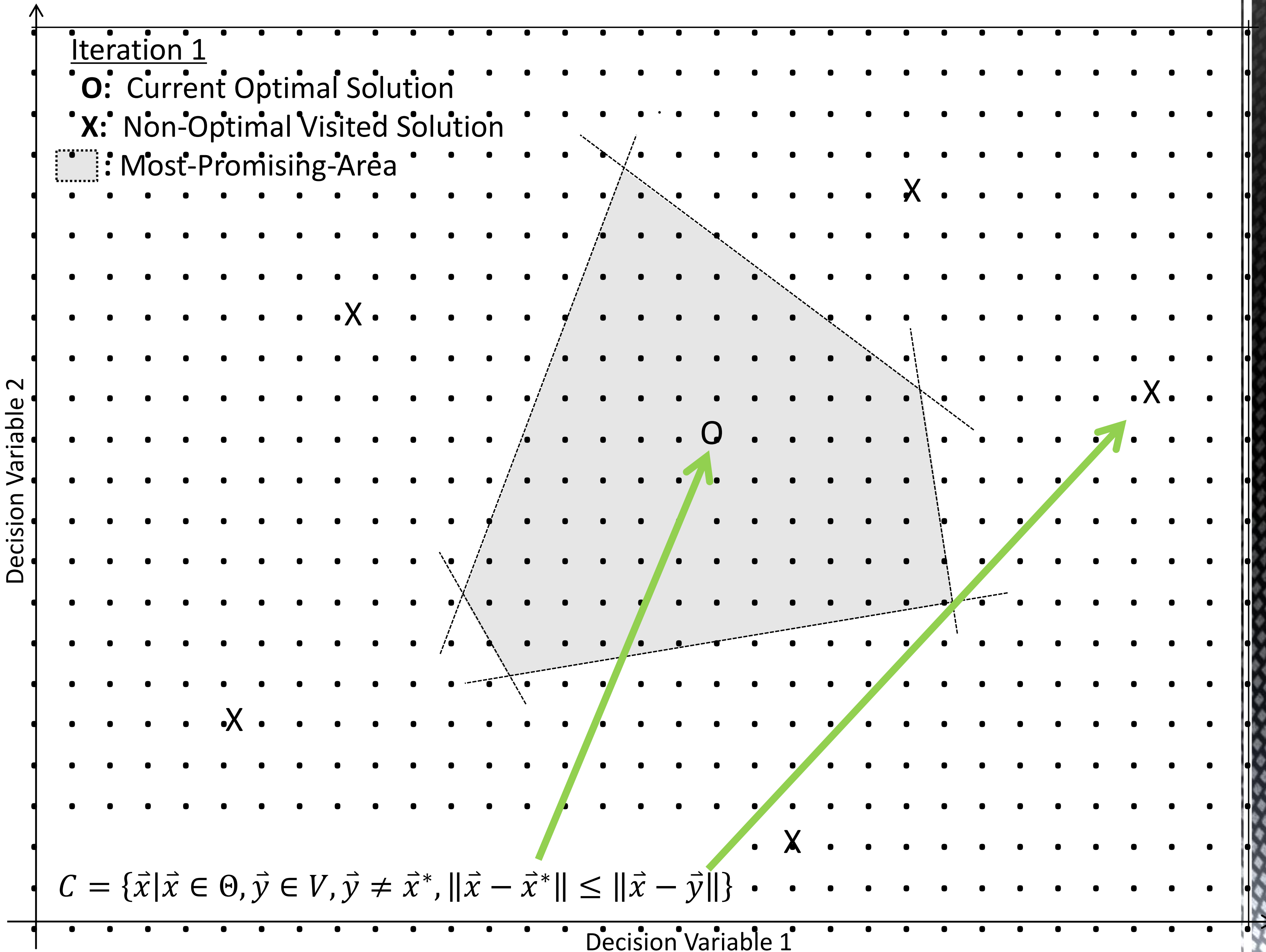
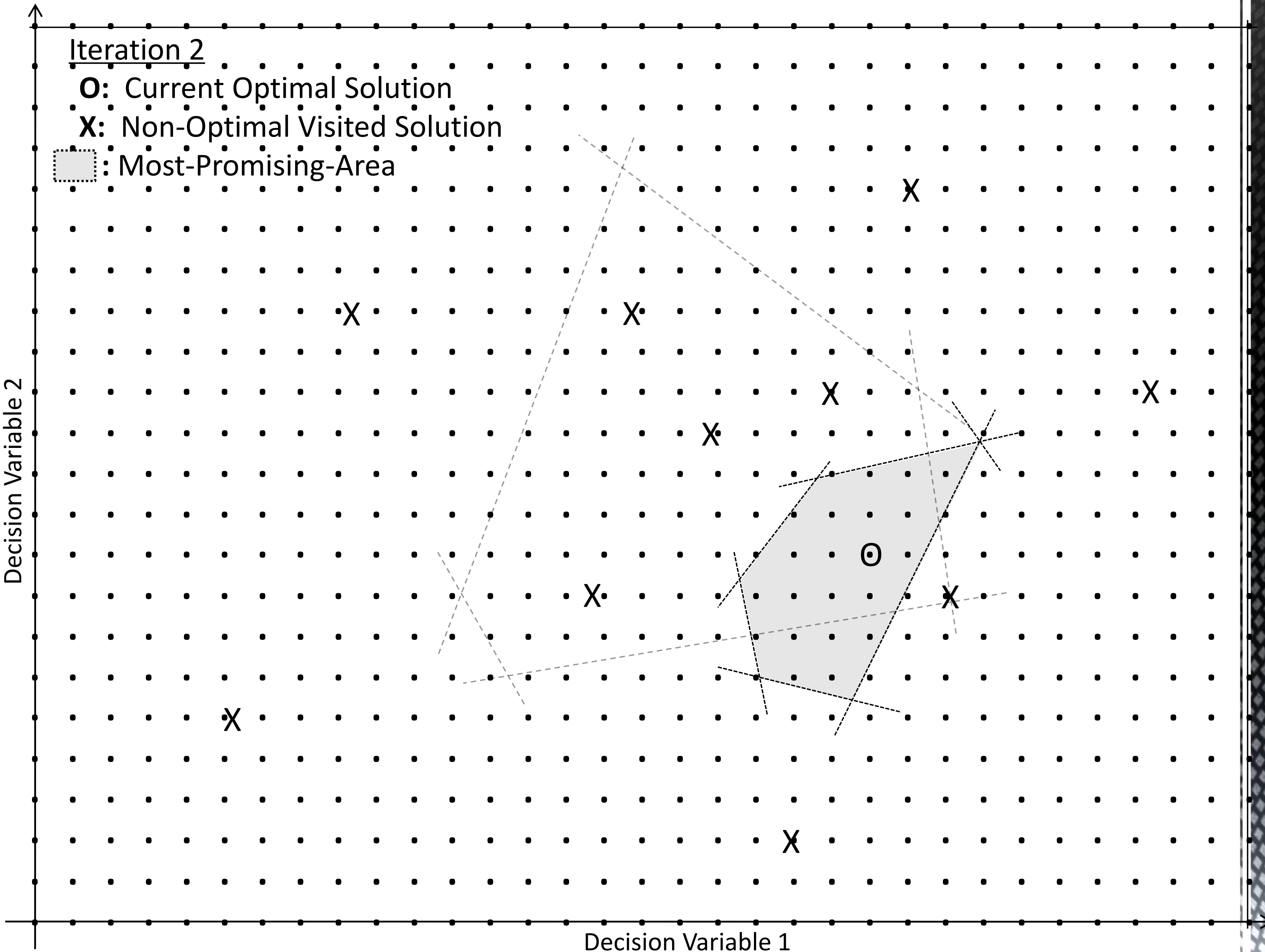
Basic Ideas of COMPASS

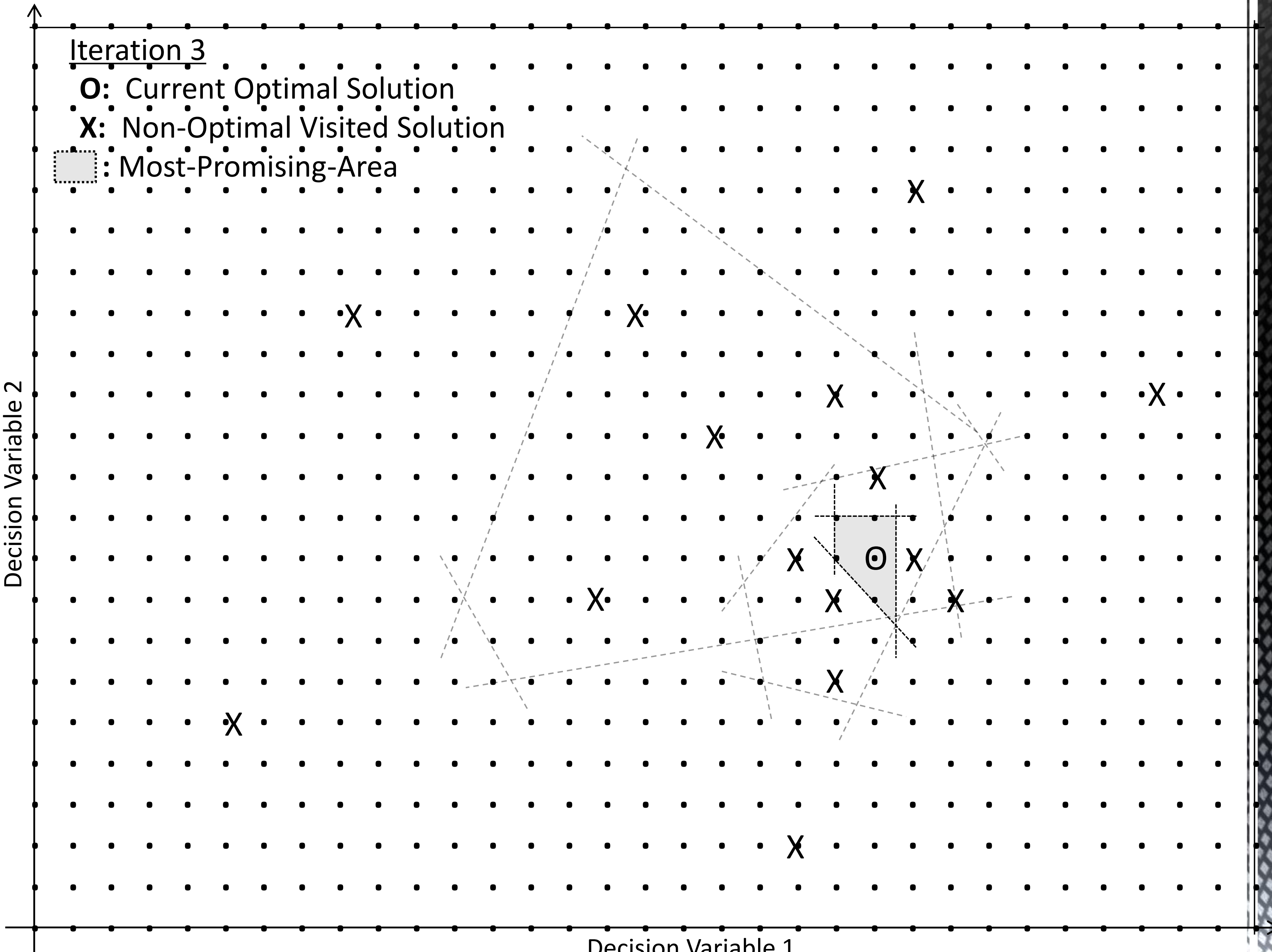
- ✓ <u>C</u>onvergent <u>O</u>ptimization via <u>M</u>ost <u>P</u>romising <u>A</u>rea <u>S</u>tochastic <u>S</u>earch, (Nelson & Hong, 2006)

- ✓ <u>Basic Assumption:</u>
  Sample in the solution space which is closer to the good solutions

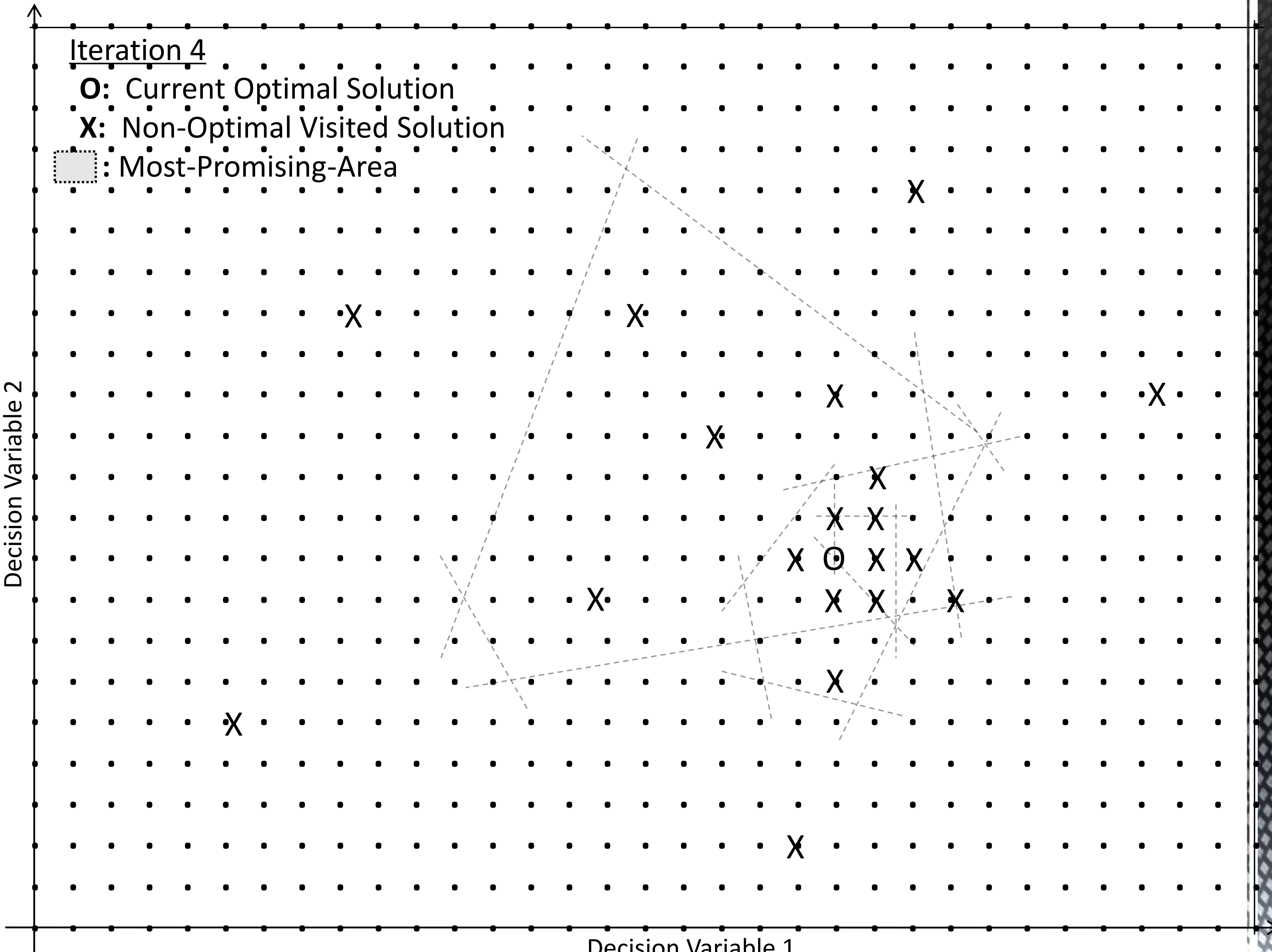



Iteration 1
O: Current Optimal Solution
X: Non-Optimal Visited Solution
: Most-Promising-Area
O
X
X
X
X
X
$C = \{\vec{x} | \vec{x} \in \Theta, \vec{y} \in V, \vec{y} \neq \vec{x}^*, \|\vec{x} - \vec{x}^*\| \leq \|\vec{x} - \vec{y}\|\}$
Decision Variable 1
Decision Variable 2


Iteration 2
O: Current Optimal Solution
X: Non-Optimal Visited Solution
: Most-Promising-Area
Decision Variable 2
Decision Variable 1

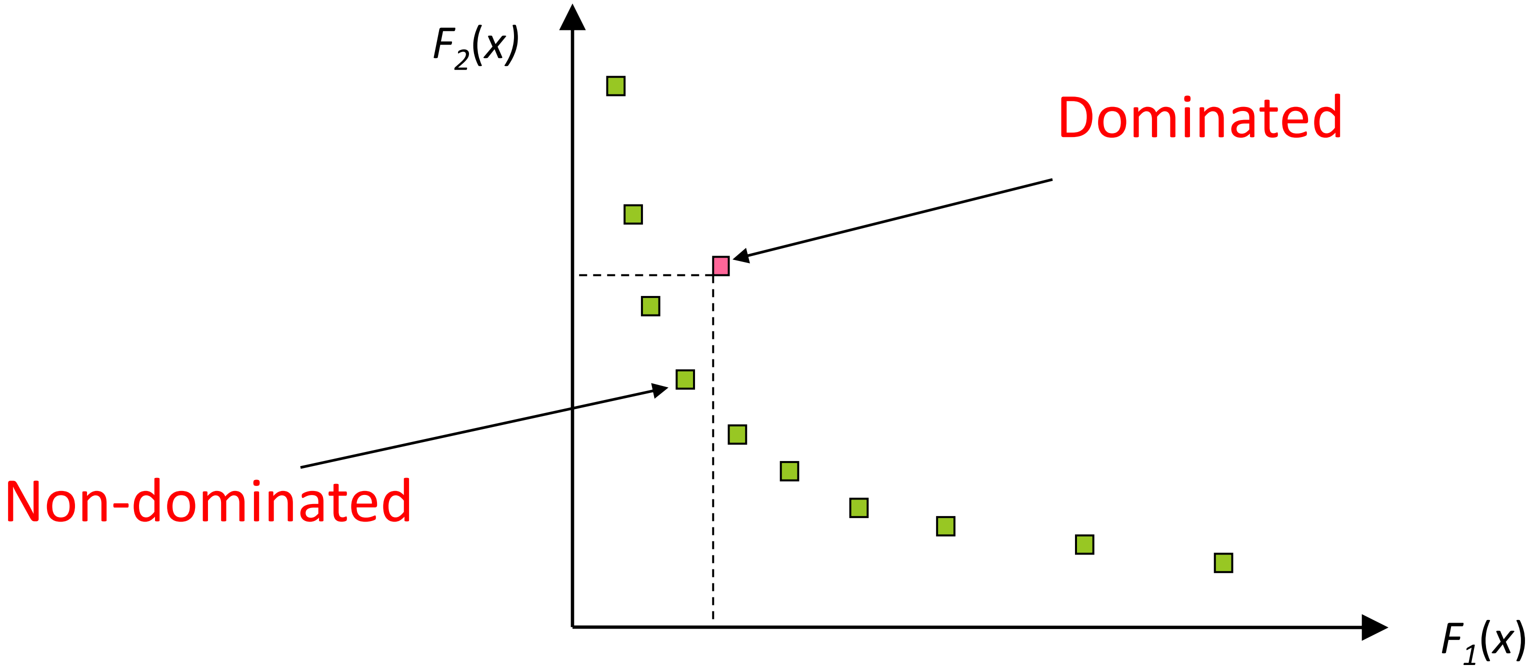


Iteration 3
O: Current Optimal Solution
X: Non-Optimal Visited Solution
: Most-Promising-Area
O
Decision Variable 1
Decision Variable 2


Iteration 4
O: Current Optimal Solution
X: Non-Optimal Visited Solution
: Most-Promising-Area
Decision Variable 2
Decision Variable 1

Definition of Pareto Optimal

- The solution to a multi-objective problem is not one solution, but a set of non-dominated solutions known as the efficient frontier (Pareto front)



Consider Minimization

Definition of Pareto Optimal

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Definition 1: Dominance (Minimizing All Objectives)
A solution $\vec{x}$ with multiple objective values $\vec{g}(\vec{x})$ is said be to dominated by another solution $\vec{y}$ with objective values $\vec{g}(\vec{y})$

i.e.

$$\vec{g}(\vec{y}) \prec \vec{g}(\vec{x}),$$

if and only if

$$\forall i \quad g_i(\vec{y}) \leq g_i(\vec{x}),$$

and

$$\exists i \quad g_i(\vec{y}) < g_i(\vec{x}).$$

Definition of Pareto Optimal

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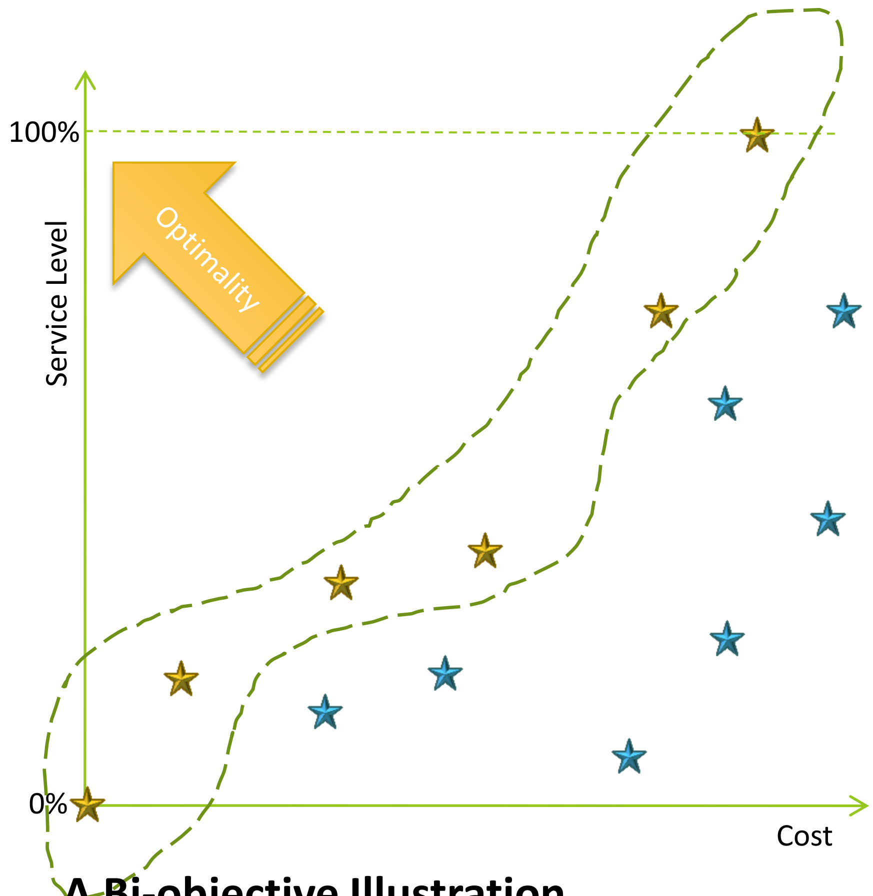
Definition 2: Pareto Optimality
A solution set $\Pi^* \subseteq \Theta$ is said to be the Pareto Optimal Set on Θ if and only if it contains all solutions in Θ that not dominated by others.

Or in other words,

$$\Pi^* = \{\vec{x} \in \Theta | \nexists \vec{y} \in \Theta, \vec{g}(\vec{y}) \prec \vec{g}(\vec{x})\}.$$

Definition of Pareto Optimal



A Bi-objective Illustration

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Definition of Local Pareto Optimal

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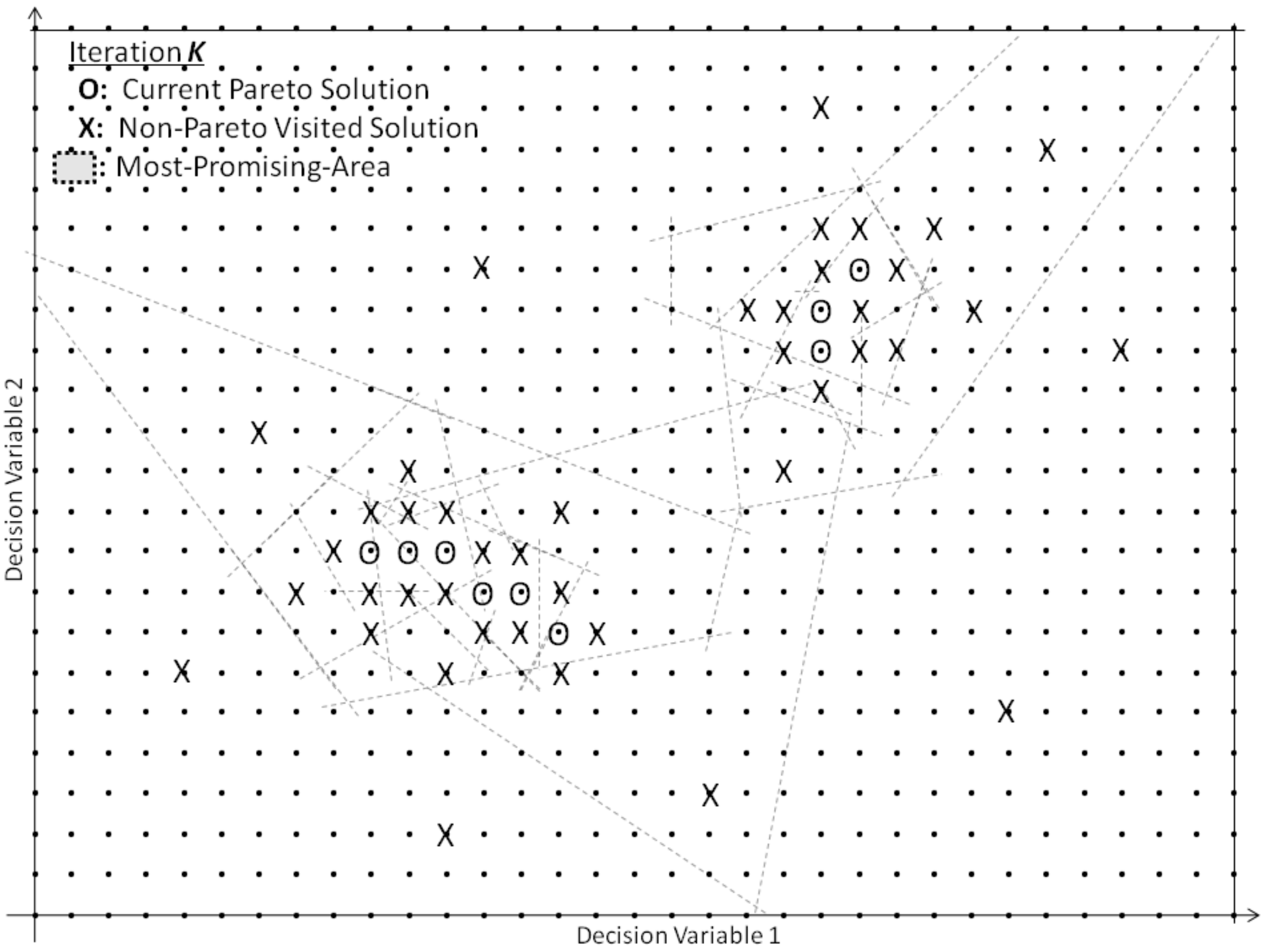
Definition 3: Local Pareto Optimality
A solution set $\mathrm{P} \subseteq \Theta$ is claimed to be a local Pareto set (LPS) on Θ if and only if no solutions in P is dominated by its neighbors that have unit Euclidean distance to it.

Or in other words,

$$\mathrm{P} \subseteq \{\vec{x} \in \Theta |\ \nexists \vec{y} \in \Theta, \|\vec{x} - \vec{y}\| \leq 1, \vec{g}(\vec{y}) \prec \vec{g}(\vec{x})\}.$$

neighbor

From Single to Multiple Objectives



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MO-COMPASS Procedure

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Step 1: Let the iteration count $k = 0$, the most-promising-area $C_0 = \Theta$, and $V_0 = \emptyset$ indicates the set of all visited solution.

Step 2: Let $k = k + 1$, sample a set of solutions X_k from C_{k-1} with batch size $|X_k| = m$ according to sampling scheme.

Step 3: Let $V_k = V_{k-1} \cup X_k$. For all $\vec{x} \in V_k$, apply SAR to collect simulation observations and identify the estimated Pareto set $\widehat{\Pi}_k$ in terms of

$$\widehat{\Pi}_k = \left\{ \forall \vec{x} \in V_k \middle| \nexists \vec{y} \in V_k, \bar{\vec{G}}(\vec{y}) \prec \bar{\vec{G}}(\vec{x}) \right\}$$

in which $\bar{\vec{G}}(\vec{x})$ is the simulation estimation of $\vec{g}(\vec{x})$.

Step 4: Construct most-promising-area at iteration k as following and repeat from Step 2.

$$C_k = \bigcup_{\vec{z} \in \widehat{\Pi}_k} \left\{ \forall \vec{x} \in \Theta \middle| \forall \vec{y} \in V_k \backslash \widehat{\Pi}_k, \|\vec{x} - \vec{z}\| \le \|\vec{x} - \vec{y}\| \right\}.$$

Promising area around the Pareto solution

Numerical Examples

Compare with NSGA-II

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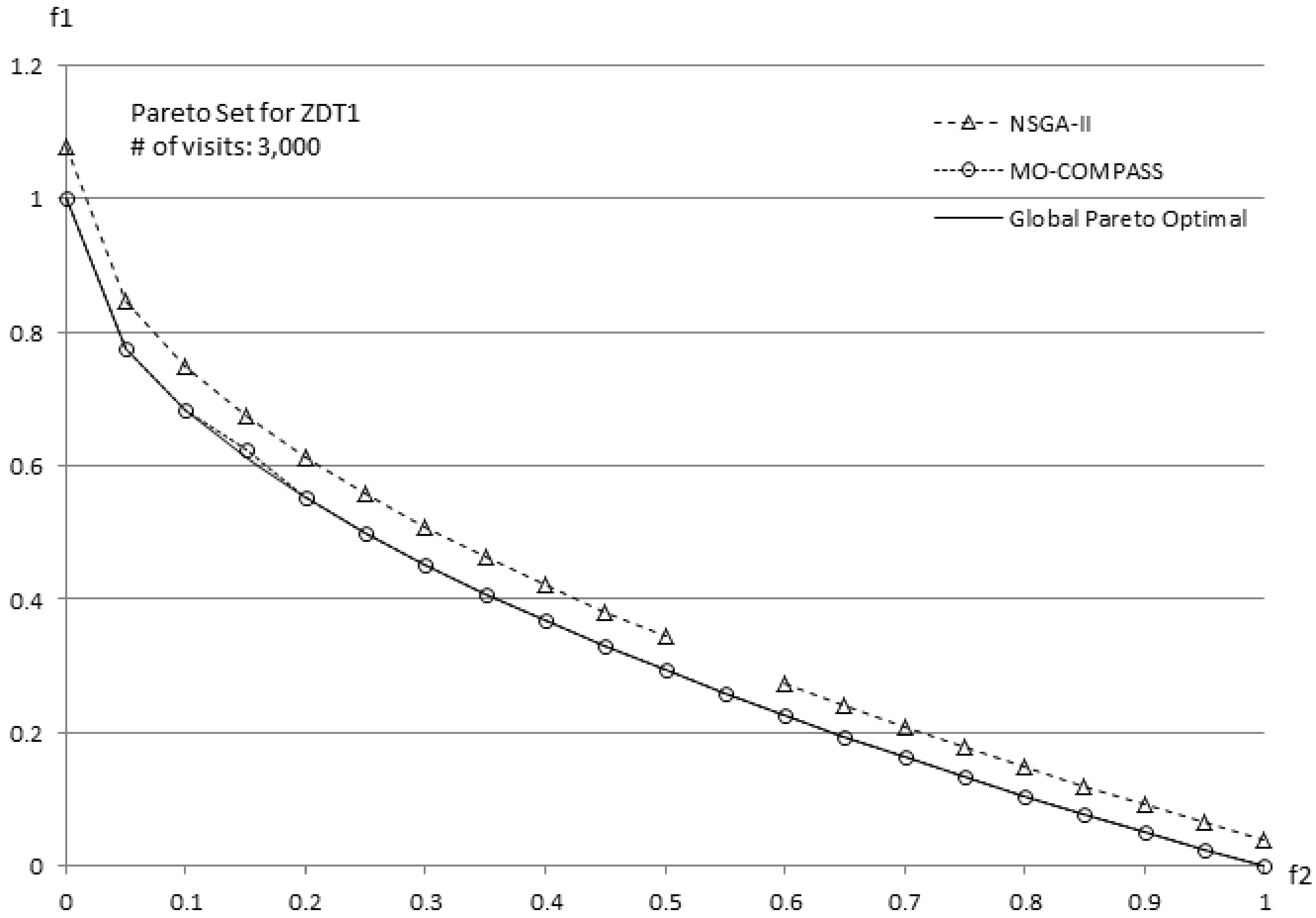
Sampling Scheme:
Coordinate Sampling (Hong, Nelson & Xu, 2010) at each $\vec{z} \in \widehat{\Pi}_k$.

SAR: Equal allocation, and let $\vec{G}(\vec{x}) = \vec{g}(\vec{x})$.

Test Problems:
Discretized ZDT1, 2, 3, 4, 6 (Zitzler, Deb and Thiele, 2000)

In the comparison, we control the maximum number of solutions that can be visited, compare $\widehat{\Pi}_k$ reached at termination by each algorithm with the global Pareto set Π^*.

MO-COMPASS vs. NSGA-II on ZDT1 (Dimension 30, Discretize Level 20)



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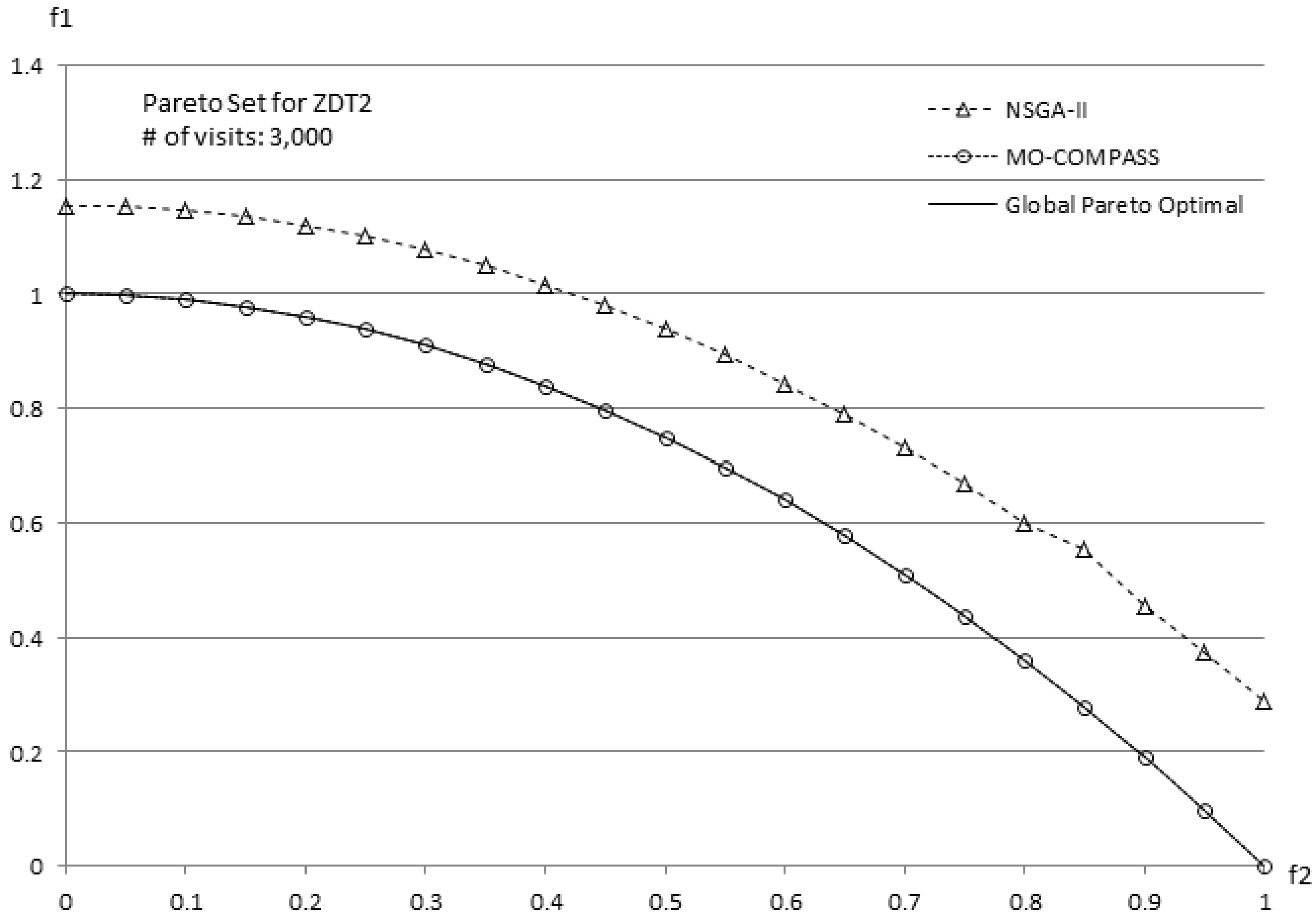
MO-COMPASS vs. NSGA-II on ZDT2 (Dimension 30, Discretize Level 20)



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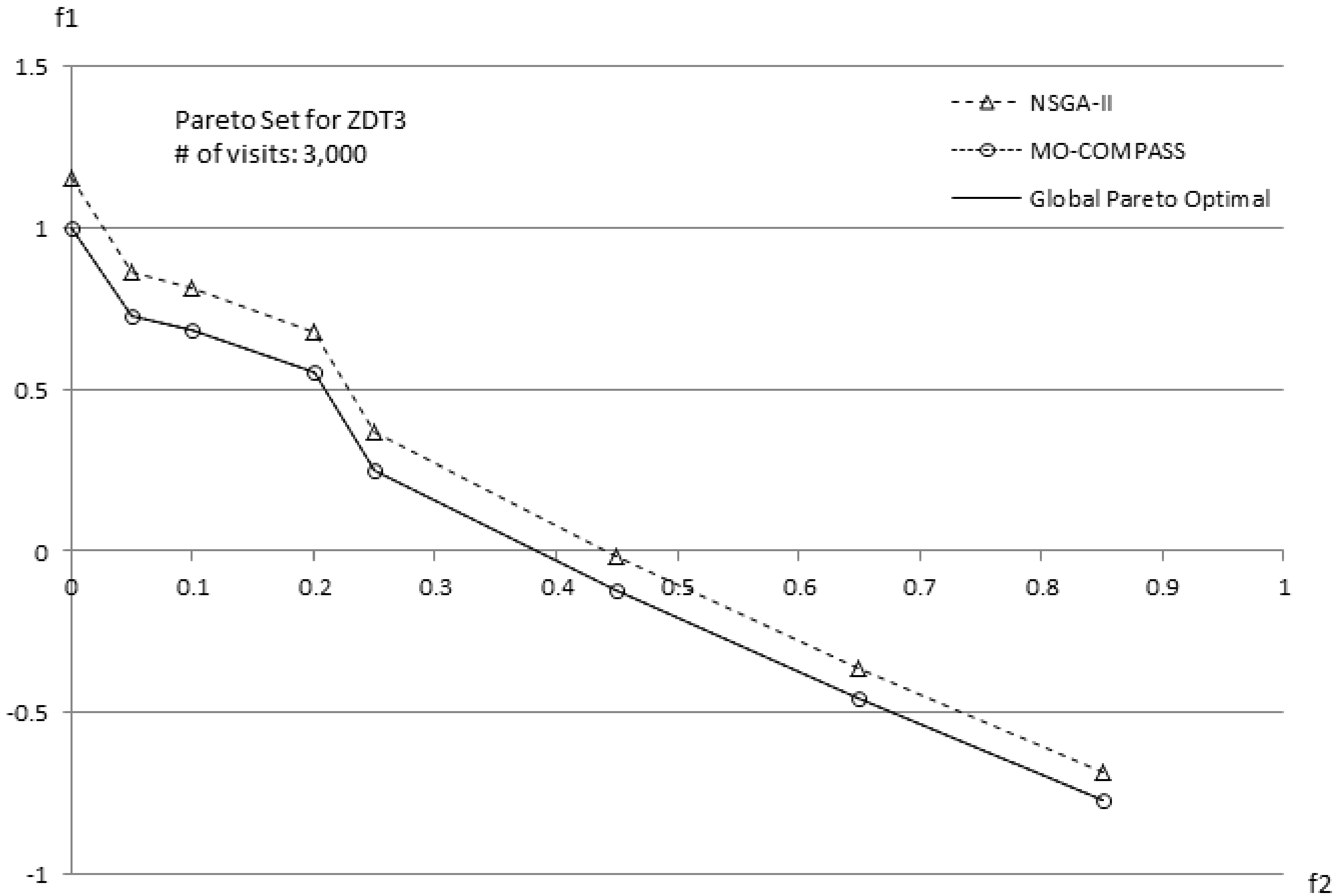
MO-COMPASS vs. NSGA-II on ZDT3 (Dimension 30, Discretize Level 20)



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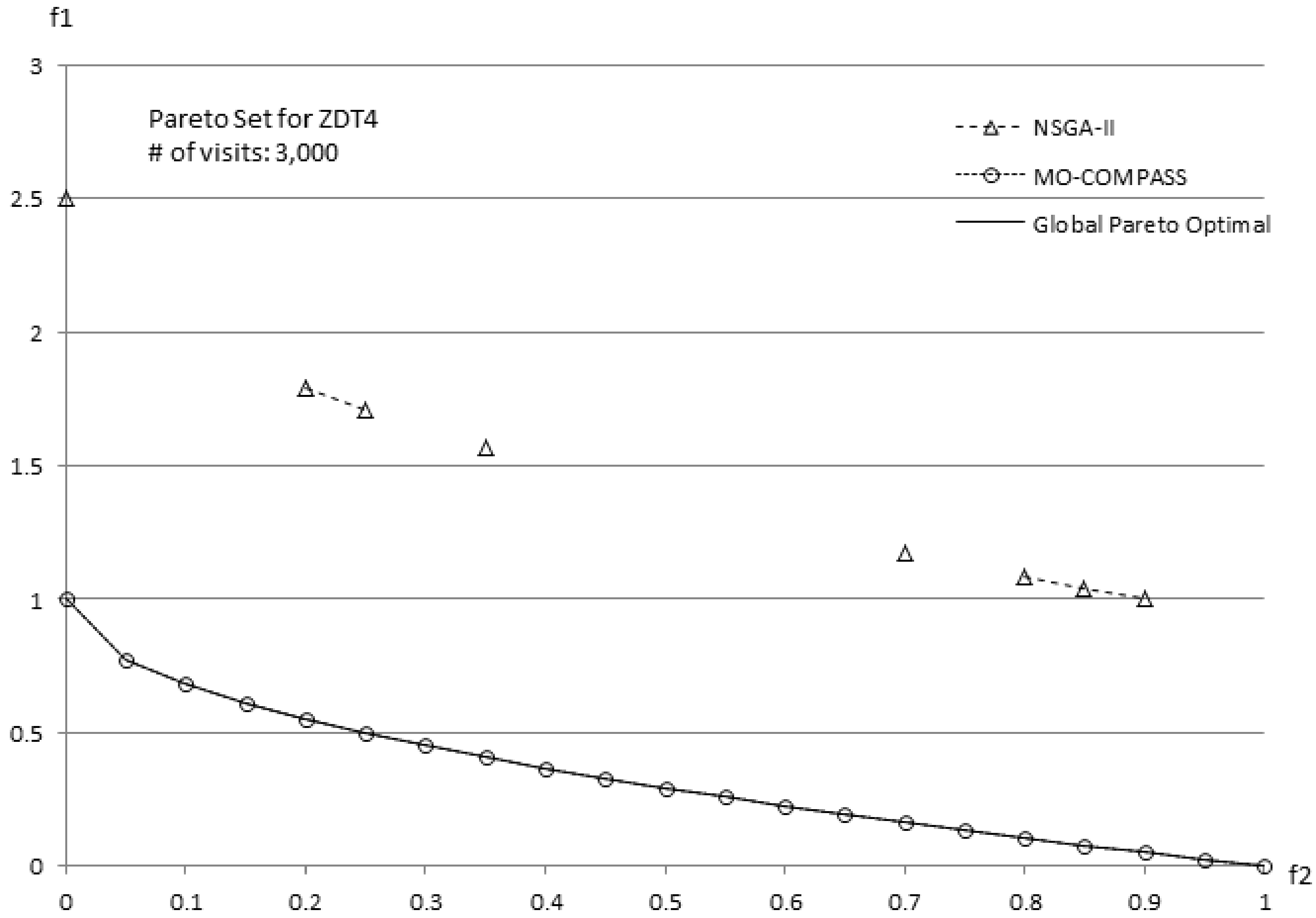
MO-COMPASS vs. NSGA-II on ZDT4 (Dimension 30, Discretize Level 20)



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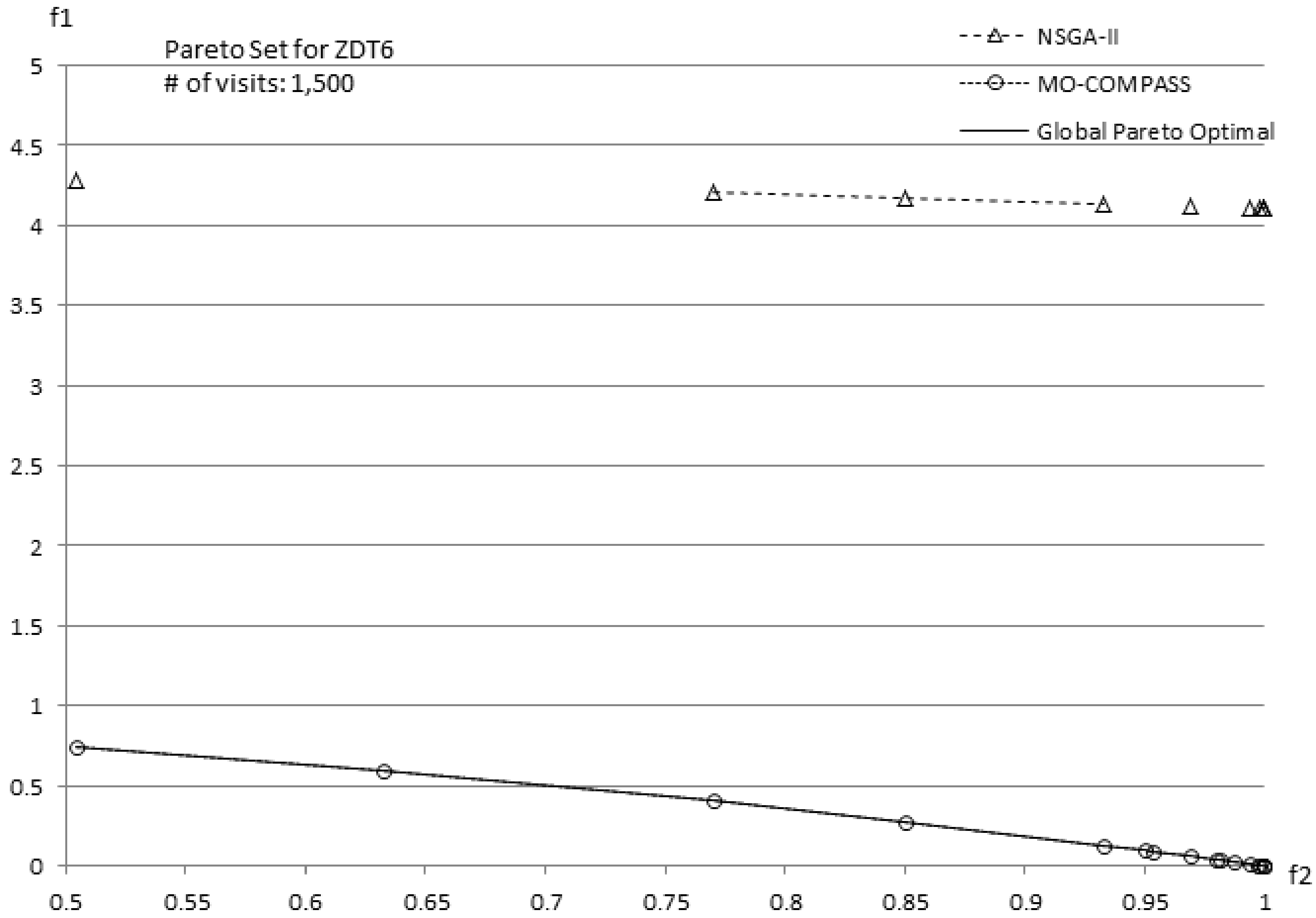
MO-COMPASS vs. NSGA-II on ZDT6 (Dimension 30, Discretize Level 20)



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Implementation in D-SIMSPAIR™

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Sampling Scheme:
Revised Mix-D (Hong & Nelson, 2006) at each $\vec{z} \in \widehat{\Pi}_k$.

SAR:
Equal allocation, and let run length to be long enough such that $\vec{G}(\vec{x}) = \vec{g}(\vec{x})$ can be assumed with small error.

Test Problems:
In the experiment, we apply MO-COMPASS to solve a D-SIMSPAIR™ Enumeration Phase problem where 5 pool stock locations are considered. The current expert heuristic enumerates 188,109 solutions in order to find the Pareto optimal set.

Implementation in D-SIMSPAIR™

Six trials of MO-COMPASS initiated with different random seeds show result as follows:

Random Seed	Number of Solutions visited before reaching CLPS		
	Expert Heuristics	MO-COMPASS	MO-COMPASS & Expert Heuristics
0	188,109	642	428
1		1060	297
2		985	284
3		955	353
4		752	195
5		1032	374
Average	188,109 (100%)	904 (0.48%)	322 (0.18%)

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Summary

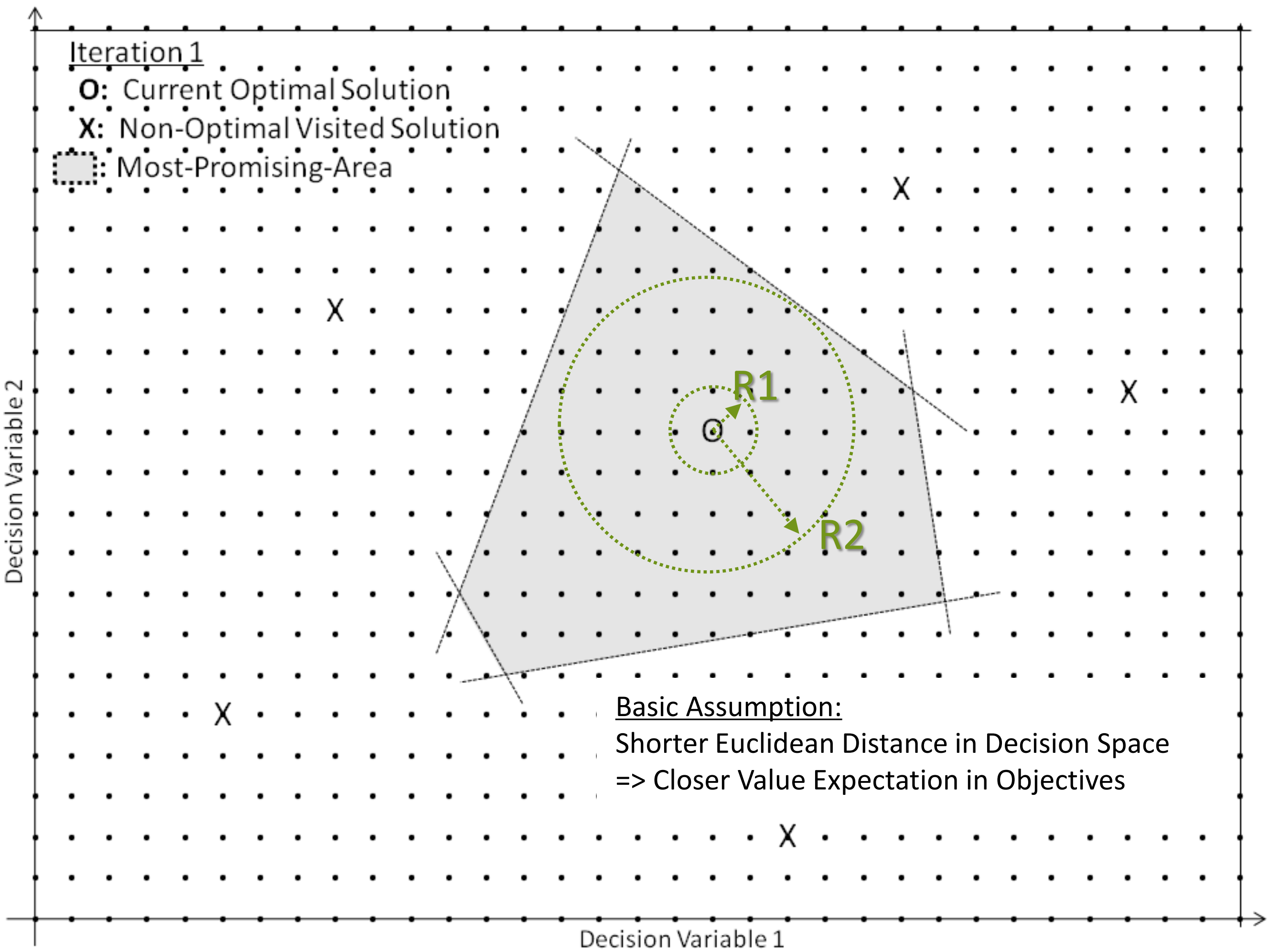
- MO-Compass is a promising search algorithm for multi-objective problem
- Sampling Algorithm affect the convergence of solutions
  - Coordinate Sampling
  - Mix-D Sampling
  - Uniform Sampling

Some Research Questions

- Sampling Algorithm
  - Random vs Information (Gradient)
  - Potential for Polar coordinate Search
- Development of SAR algorithm
- Structure of Multi-objective Problem
  - Regularity
- Development of an Integrated Approach for D-SIMSPAIR

Polar Coordinate Sampling- (Polar Search)

Weakness of Uniform Sampling



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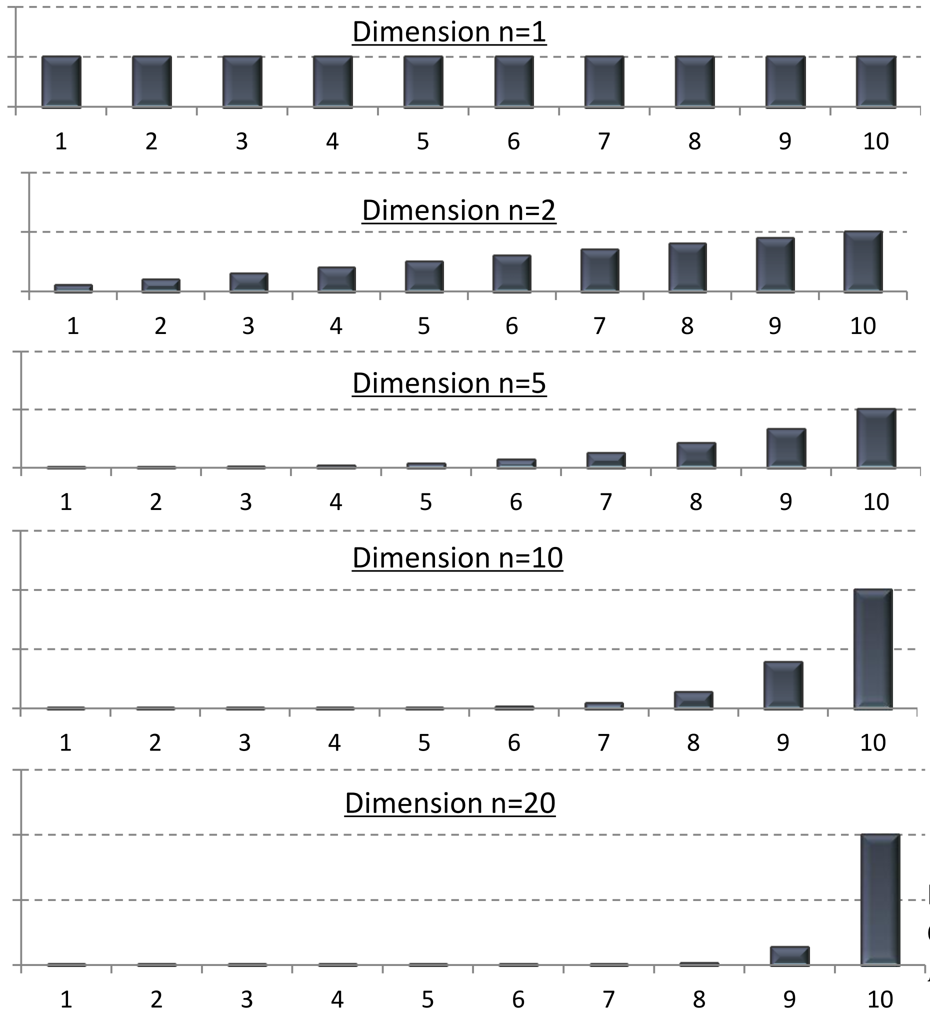
Weakness of Uniform Sampling

Sampling Density

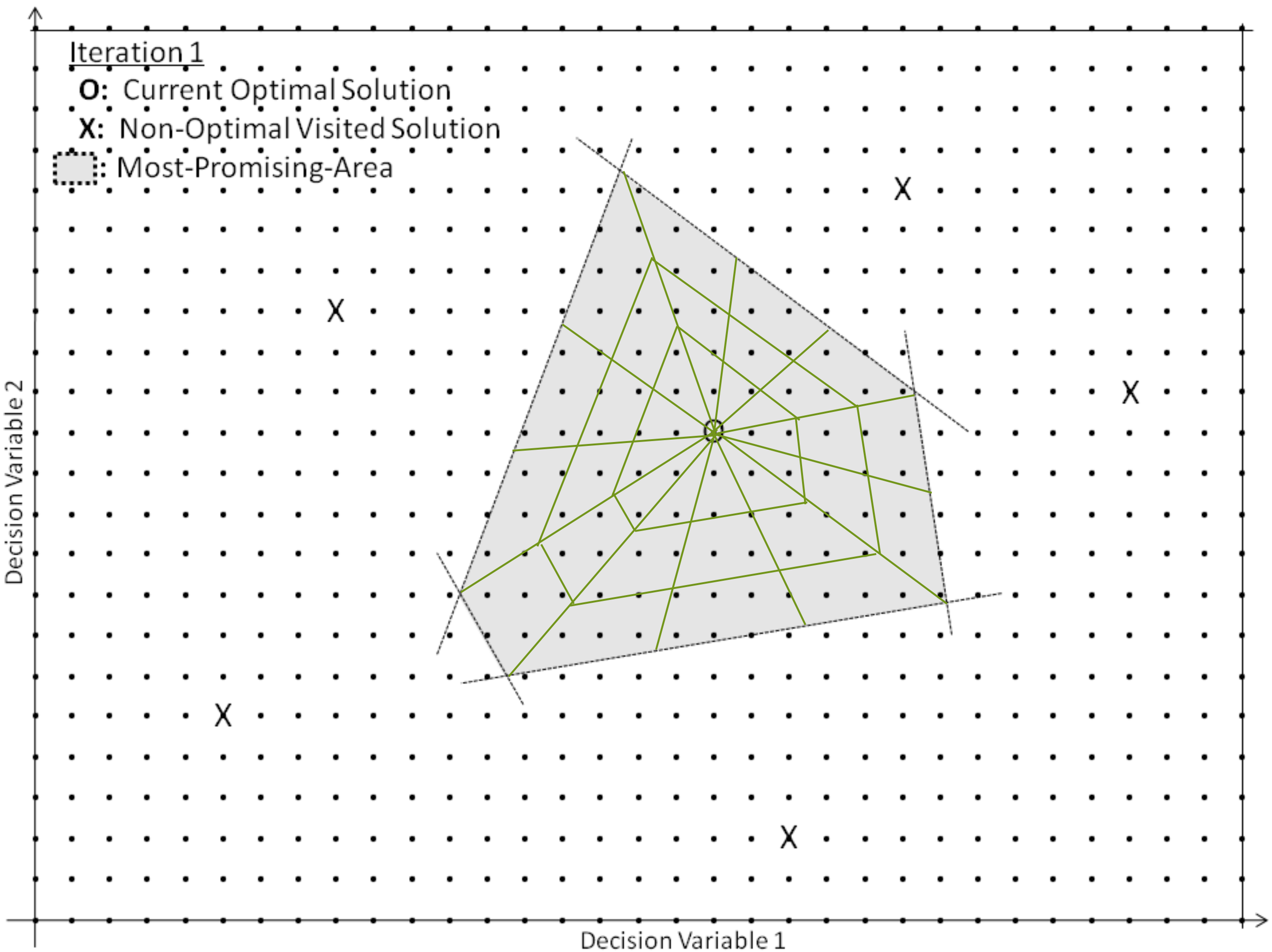
Surface on a hypersphere:

$$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}R^n$$



Distance from Current Optimal R

New Sampling Scheme: Polar Distribution



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3-D Polar Coordinates

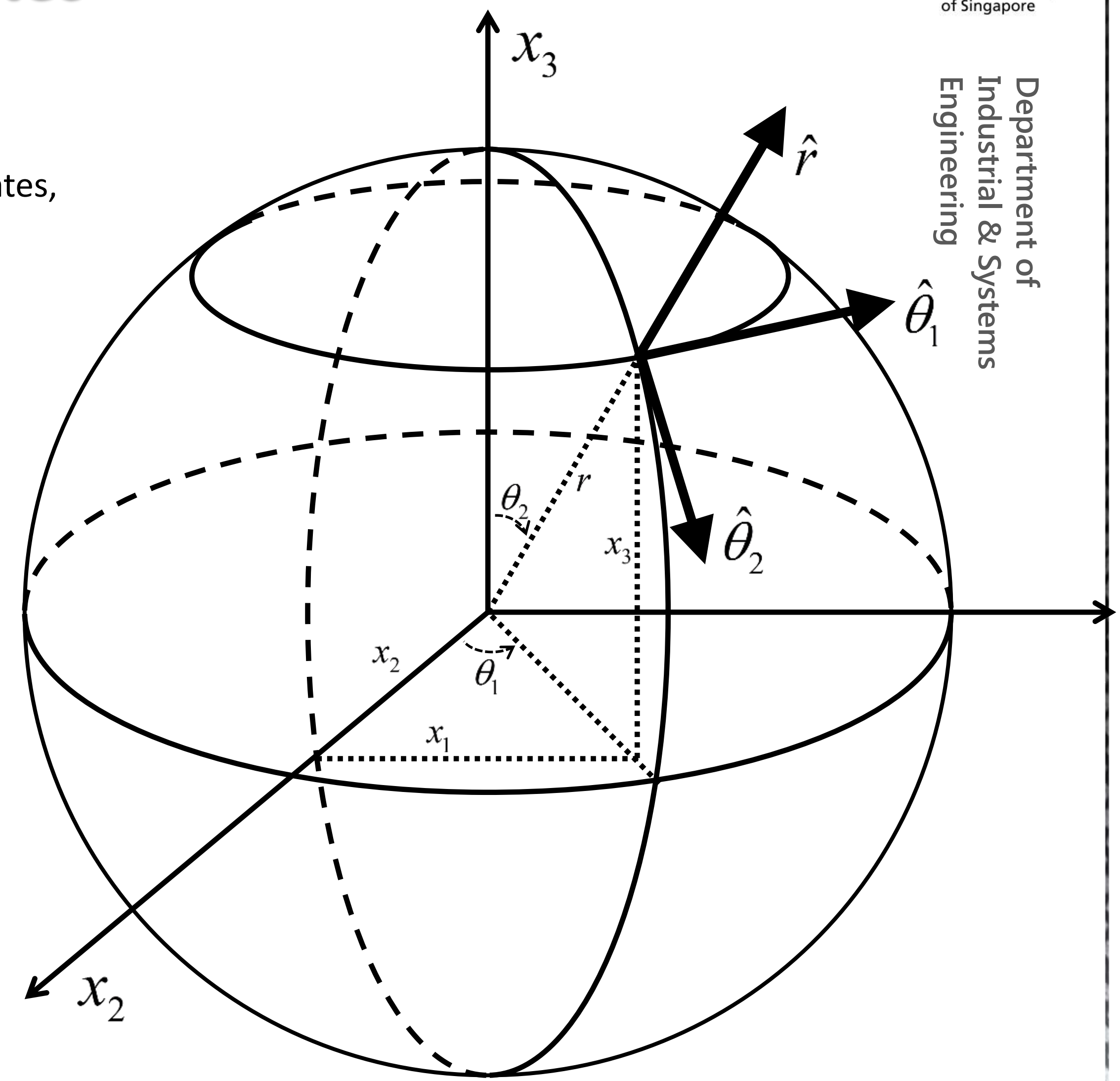
- 3-D Polar Coordinates:
  - Often referred as Spherical Coordinates, or Spherical Polar Coordinates
  - Consist of
    - 2 Angular Coordinate
    - 1 Radial Coordinate

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$x_1 = r \sin\theta_1 \sin\theta_2$$

$$x_2 = r \cos\theta_1 \sin\theta_2$$

$$x_3 = r \cos\theta_2$$



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p-D Polar Coordinates

- p-D Polar Coordinates (Generalized):
  - Seldom discussed in literature
  - Consist of
    - p-1 Angular Coordinate
    - 1 Radial Coordinate

- Definition:
  - *In a p-dimensional polar coordinate system, a point is denoted by $[r,\boldsymbol{\theta}]$, in which $r \in [0,\infty)$ and $\boldsymbol{\theta} \in [0,2\pi) \times [0,\pi]^{p-2}$, if its Euclidean distance from the origin is r (radial coordinate) and $\boldsymbol{\theta}$ (angular coordinate) refers its direction in the space in the sense that θ_i denotes its angle with respect to the positive direction of the $i+1^{th}$ axis towards the hyperplane spanned by the first i axes.*

p-D

$$r = \sqrt{\sum_{i=1}^{p} x_i^{\,2}}$$

$$x_1 = r \prod_{j=1}^{p-1} \sin\theta_j$$

$$x_i = r\cos\theta_{i-1} \prod_{j=i}^{p-1} \sin\theta_j \quad \text{for } 2 \le i \le p$$

2-D

$$r = \sqrt{x_1^2 + x_2^2}$$

$$x_1 = r\sin\theta_1$$

$$x_2 = r\cos\theta_1$$

3-D

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$x_1 = r\sin\theta_1\sin\theta_2$$

$$x_2 = r\cos\theta_1\sin\theta_2$$

$$x_3 = r\cos\theta_2$$

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Polar Coordinate

- Can generate uniform direction
- Can generate a concentrated distribution on a certain direction (gradient direction)
- Random Gradient Search
  - Random Local Search + Gradient Search
  - GO-Polar (The Gradient-Oriented Polar Random Search)

Polar Uniform Distribution

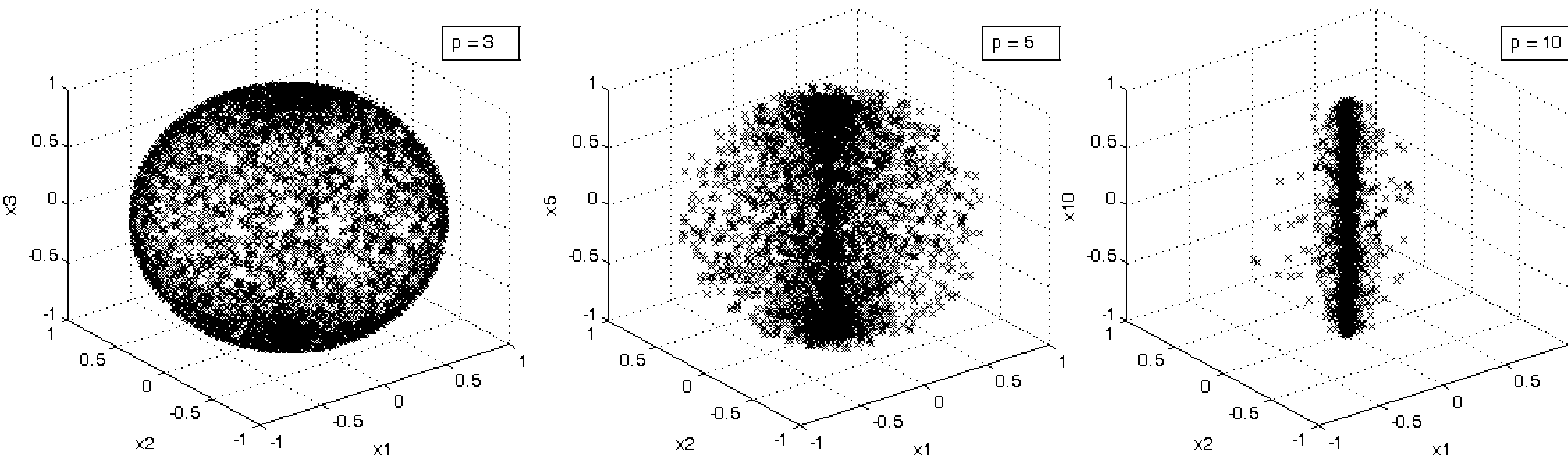
- A simplest random distribution on directions
- A Trial:
  - ✓ Draw each θ_i uniformly on its domain
  - ✓ Keep $r = 1$

p-D

$$r = \sqrt{\sum_{i=1}^{p} x_i^{\,2}}$$

$$x_1 = r\prod_{j=1}^{p-1}\sin\theta_j$$

$$x_i = r\cos\theta_{i-1}\prod_{j=i}^{p-1}\sin\theta_j \quad \text{for} \;\; 2 \le i \le p$$



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Polar Uniform Distribution

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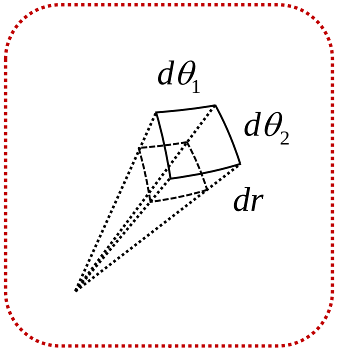
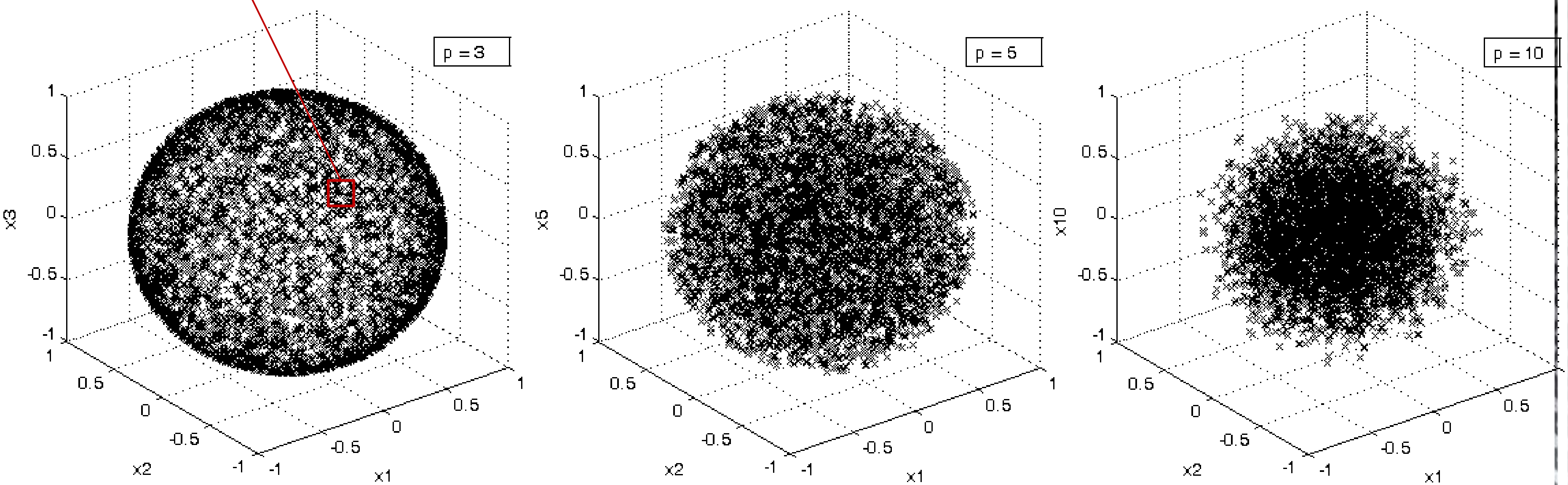


$$\frac{f(r,\boldsymbol{\theta})\ dr \cdot d\theta_1 \cdots d\theta_{p-1}}{dx_1 \cdots dx_p} = c$$

$$\Rightarrow f(r,\boldsymbol{\theta}) \propto J_{r,\boldsymbol{\theta}} = \frac{dx_1 \cdots dx_p}{dr \cdot d\theta_1 \cdots d\theta_{p-1}} = r^{p-1}\prod_{j=1}^{p-1}\sin^{j-1}\theta_j$$

$$\Rightarrow f_i(\theta_i) \propto \sin^{i-1}\theta_i$$



Concentrated Polar Distribution

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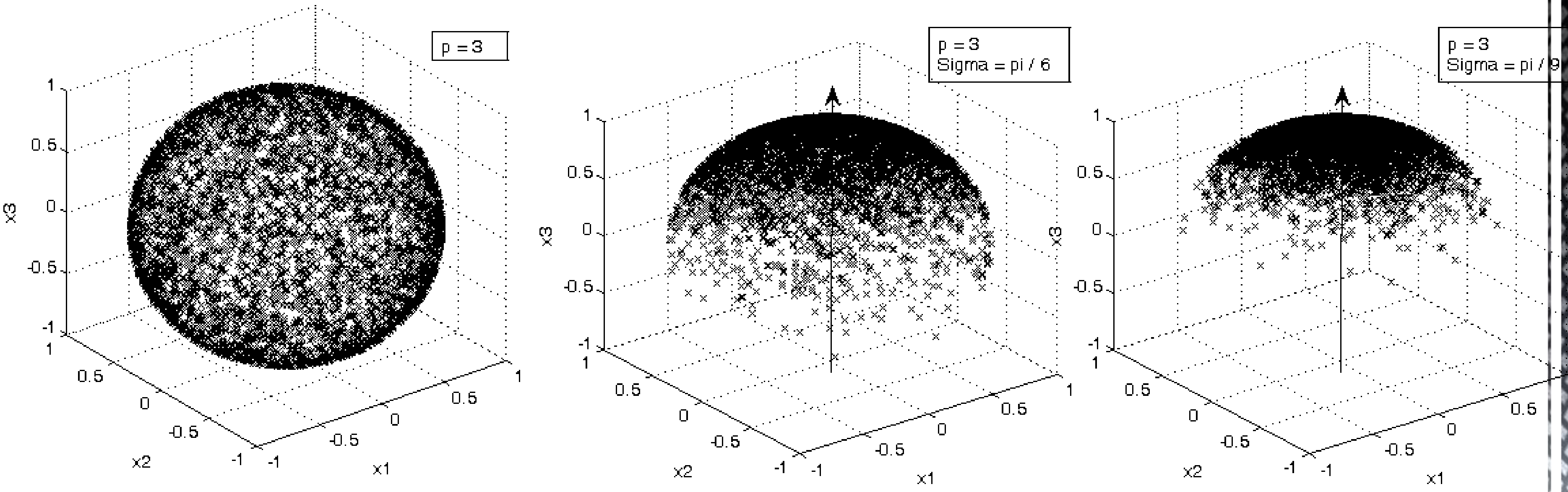
Let $r = 1$,

$$x_1 = \sin\theta_1 \qquad x_2 = \cos\theta_1$$

$$x_1 = \sin\theta_1 \sin\theta_2 \qquad x_2 = \cos\theta_1 \sin\theta_2 \qquad x_3 = \cos\theta_2$$

$$x_1 = \sin\theta_1 \sin\theta_2 \sin\theta_3 \qquad x_2 = \cos\theta_1 \sin\theta_2 \sin\theta_3 \qquad x_3 = \cos\theta_2 \sin\theta_3 \qquad x_4 = \cos\theta_3$$

- Easy to rise up to higher dimension
- Limited Disturbance



Polar "Normal" Distribution

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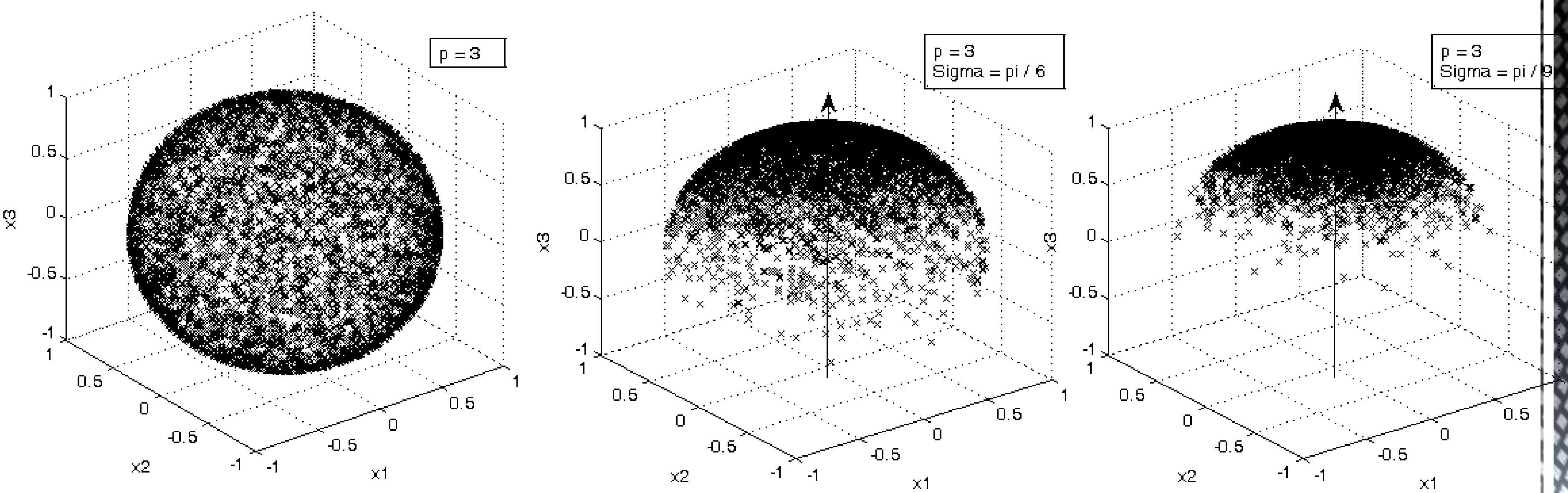
$$f_{i-1}(\theta_{i-1}) \propto \sin^{i-2}\theta_{i-1}, \quad \text{for } 1 \le i < p-1$$

$$f_{p-1}(\theta_{p-1}) \propto \sin^{p-2}\theta_{p-1} \cdot \phi_{0,\sigma^2}(\theta_{p-1})$$



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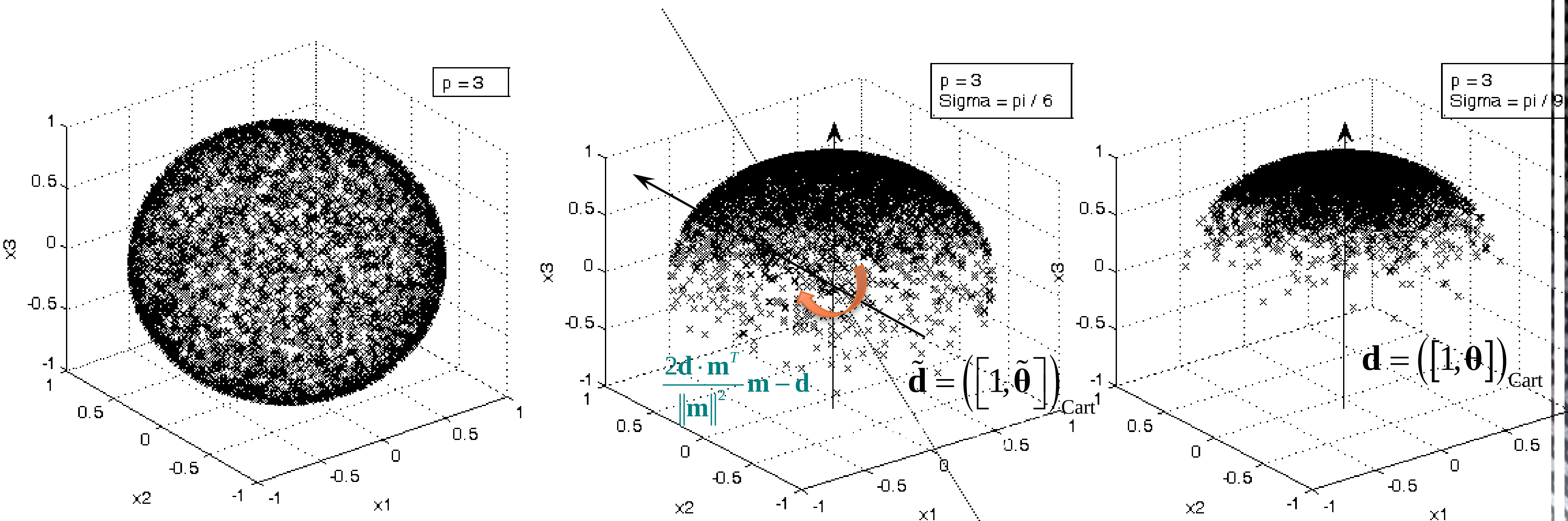
Polar “Normal” Distribution

$$f_{i-1}\left(\theta_{i-1}\right) \propto \sin^{i-2}\theta_{i-1}, \quad \text{for } 1 \le i < p-1$$

$$f_{p-1}\left(\theta_{p-1}\right) \propto \sin^{p-2}\theta_{p-1} \cdot \phi_{0,\sigma^2}\left(\theta_{p-1}\right)$$

Linear Transformation

$$\boldsymbol{\theta} \sim N^{p}_{\text{polar}}\left(\tilde{\boldsymbol{\theta}}, \sigma^2\right)$$



GO-POLARS

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- The Gradient-Oriented Polar Random Search:
- *Step 1:* (Initialization) Pick an initial guess $\hat{\mathbf{x}}_0 \in \Theta$, and set $k = 0$.
- *Step 2:* Generate a direction $\mathbf{d}_k$ from $\mathrm{N}_{\text{polar}}^{p}\left(\mathbf{g}\left(\hat{\mathbf{x}}_k\right), \sigma_k^2\right)$ in which $\mathbf{g}\left(\hat{\mathbf{x}}_k\right)$ is the gradient at $\hat{\mathbf{x}}_k$.
- *Step 3:* Let $\hat{\mathbf{x}}_{\text{new}} = \hat{\mathbf{x}}_k - a_k \left\|\mathbf{g}\left(\hat{\mathbf{x}}_k\right)\right\| \mathbf{d}_k$. If $\hat{\mathbf{x}}_{\text{new}} \in \Theta$ and $L\left(\hat{\mathbf{x}}_{\text{new}}\right) < L\left(\hat{\mathbf{x}}_k\right)$, set $\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_{\text{new}}$, otherwise $\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k$.
- *Step 4:* Set $k = k + 1$. Go to *Step 2*.

- The convergent property of the GO-Polar (similar to stochastic approximation)

Numerical Examples

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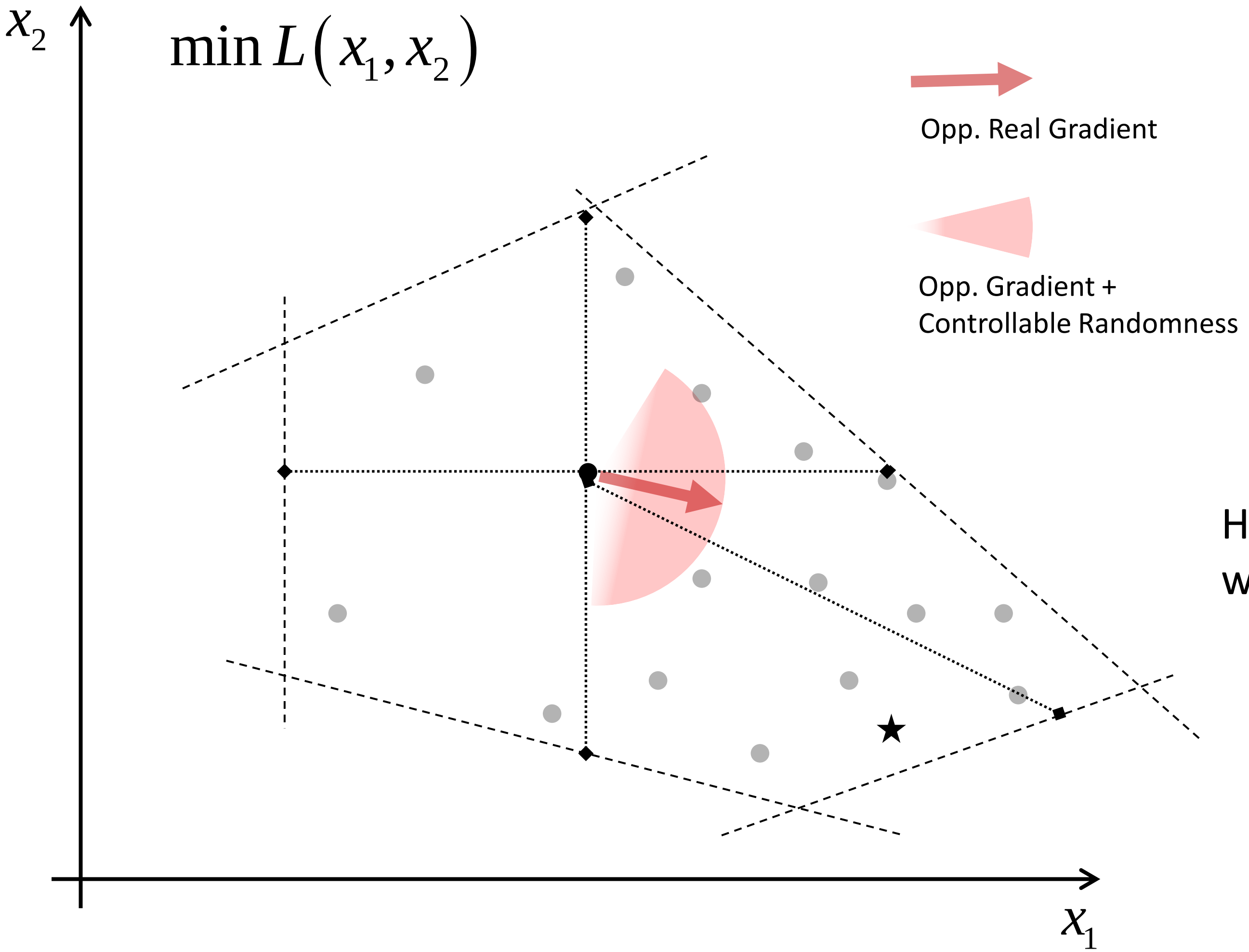
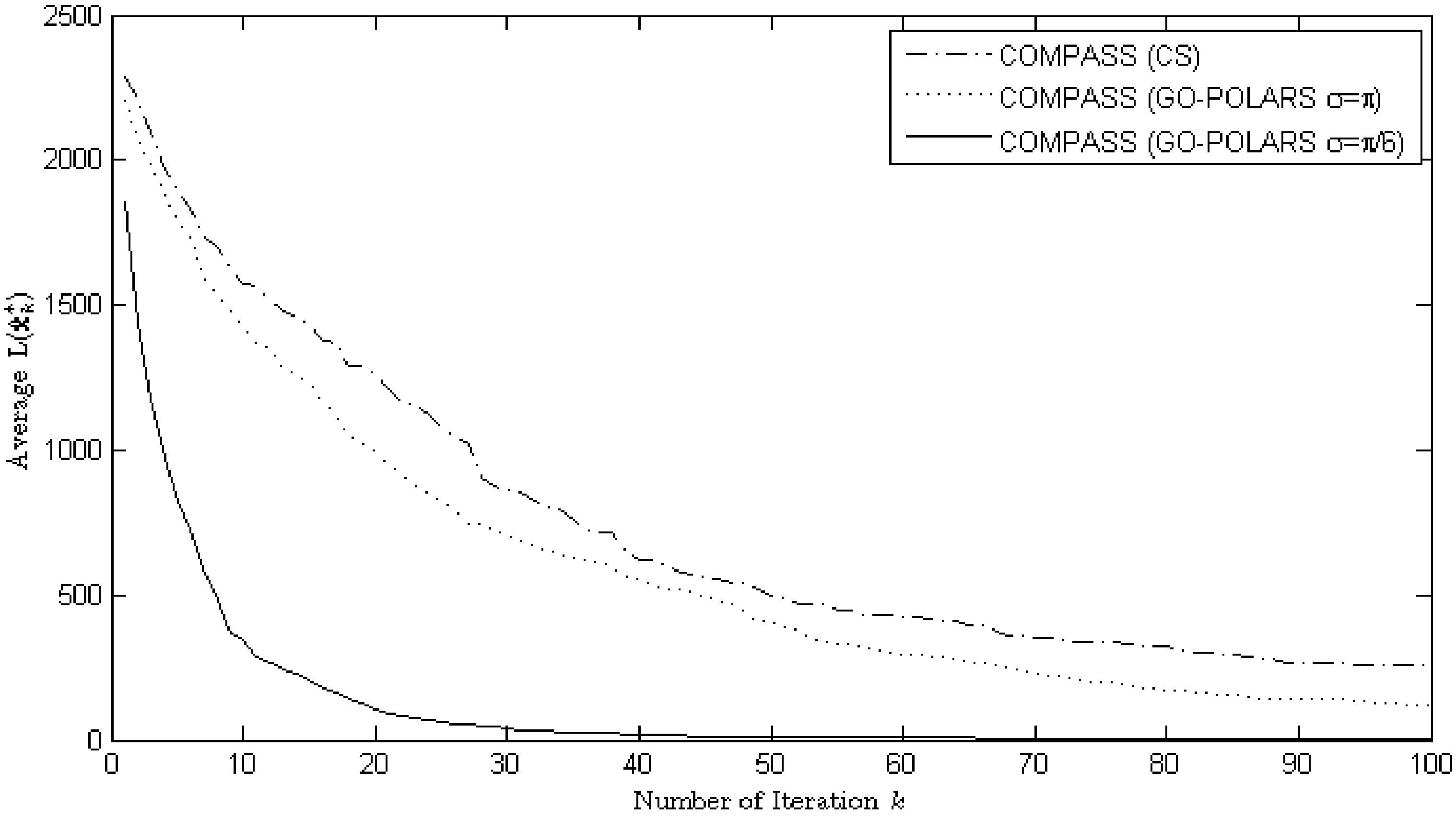


Hybridize GO-POLARS
with COMPASS

Numerical Examples

- Average $L\left(\hat{\mathbf{x}}_k^*\right)$ from 50 replications



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Conclusions

- Why shall we GO-POLARS?
  - Promising Numerical Results
  - Wide Application of Derivatives
  - Potential for Breath (exploration) vs Depth (exploitation)

- Future Study:
  - Analyze the uncertainty in the search direction (controlled random+ nature random) and search efficiency for different problem
  - Discrete vs continuos
  - Hybridize GO-POLARS with other random search algorithms