A Change-of-Variable Approach to Simulating Conditional Expectations

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Outline

1. A change-of-variable viewpoint of conditional Monte Carlo
2. Simulating conditional expectations
3. Application to Greek estimation of financial options
4. Numerical examples
5. Conclusions

Conditional Monte Carlo (CMC)

- Let $X = (X_1, \ldots, X_m)$ be an m-dimensional random vector.
- For a given function g, one may want to estimate

$$\alpha = \mathrm{E}[g(X)].$$

- Conditional Monte Carlo

$$\alpha = \mathrm{E}\left(\mathrm{E}\left[g(X)|Y\right]\right) = \mathrm{E}\left[r(Y)\right],$$

  where Y is a random vector, and $r(Y) = \mathrm{E}[g(X)|Y]$.
- Main difficulty: how to select an appropriate Y such that $r(\cdot)$ has a closed-form expression, or can be evaluated easily.

Properties of CMC

- **Variance Reduction**. Because

$$\mathrm{Var}(g(X)) = \mathrm{E}\left(\mathrm{Var}(g(X)|Y)\right) + \mathrm{Var}\left(r(Y)\right),$$

the variance of the CMC estimator $r(Y)$ is less than or equal to that of $g(X)$.

- **Smoothing**. In the case where g is discontinuous, the function obtained from CMC, $r(\cdot)$, may be continuous.
- CMC finds applications in broad areas of operations research; see the monograph by Fu and Hu (1997).

Intuitions

- Roughly speaking, CMC turns an expectation w.r.t. X to an expectation w.r.t. Y.
- It shares the same spirit as *change of variable* in elementary calculus.
- Can CMC be understood from a change-of-variable perspective?
- Hopefully this perspective may lead to some new insights.

A Motivated Example

- Let X_1, X_2 be independent exponential random variables with mean 1. Let F_i and f_i denote the cdf and pdf of X_i resp.
- For some constant K, we want to estimate

$$\alpha = \mathrm{E}\left[1_{\{X_1+X_2\le K\}}\right].$$

- CMC with conditioning on X_1 produces

$$\alpha = \mathrm{E}\left(\mathrm{E}\left[1_{\{X_1+X_2\le K\}}|X_1\right]\right) = \mathrm{E}\left[F_2(K-X_1)\right].$$

A Motivated Example (*cont.*)

A change-of-variable viewpoint ($f(\cdot,\cdot)$: the density of (X_1, X_2)):

1. Construct a 1-1 mapping: $(x_1, x_2) \to (y_1, y_2) \equiv (x_1, x_1 + x_2)$. Jacobian of the mapping is 1.

2. By change of variable,

$$\alpha = \int \int 1_{\{y_2 \le K\}} f(y_1, y_2 - y_1)\, dy_2\, dy_1 = \mathrm{E}\left[r(Y_1)\right],$$

   where $(Y_1, Y_2) = (X_1, X_1 + X_2)$,

$$r(y_1) = \frac{\int 1_{\{y_2 \le K\}} f(y_1, y_2 - y_1)\, dy_2}{\int f(y_1, y_2 - y_1)\, dy_2} = F_2(K - y_1).$$

A Motivated Example (*cont.*)

Can we construct other 1-1 mappings?

An interesting 1-1 mapping:

$$(x_1, x_2) \to (y_1, y_2, z) \equiv \left(\frac{x_1}{x_1 + x_2}, \frac{x_2}{x_1 + x_2}, x_1 + x_2 \right).$$

- Image space of the mapping: $\{(y_1, y_2, z) : y_1 + y_2 = 1, z > 0\}$. Note that domain of (X_1, X_2) is $\mathbb{R}_+^2$.
- Jacobian of the mapping: $\sqrt{2}/z$

A Motivated Example (*cont.*)

- Denote $y = (y_1, y_2)$. Let $(Y, Z) = \left(\frac{X_1}{X_1+X_2}, \frac{X_2}{X_1+X_2}, X_1 + X_2\right)$. Density of (Y, Z) is

$$\tilde{f}(y, z) = \frac{z}{\sqrt{2}} f(y_1 z, y_2 z).$$

- By change of variable,

$$\alpha = \int \int 1_{\{z \le K\}} \tilde{f}(y, z)\, dz\, dy = \mathrm{E}\left[r(Y)\right],$$

where

$$r(y) = \frac{\int 1_{\{z \le K\}} \tilde{f}(y, z)\, dz}{\int \tilde{f}(y, z)\, dz} = \alpha.$$

General Mappings

Consider the following 1-1 mapping:

$$u : x = (x_1, \ldots, x_m) \to (u_1(x), \ldots, u_n(x)).$$

Let $u^{-1}(v) = x$ if $u(x) = v$, where $v = (v_1, \ldots, v_n)$.

Derivative matrix of the mapping is defined as

$$Du(x) = \begin{pmatrix} \partial u_1(x)/\partial x_1 & \partial u_2(x)/\partial x_1 & \ldots & \partial u_n(x)/\partial x_1 \\ \partial u_1(x)/\partial x_2 & \partial u_2(x)/\partial x_2 & \ldots & \partial u_n(x)/\partial x_2 \\ \vdots & \vdots & \vdots & \vdots \\ \partial u_1(x)/\partial x_m & \partial u_2(x)/\partial x_m & \ldots & \partial u_n(x)/\partial x_m \end{pmatrix}.$$

General Mappings (*cont.*)

Let $J(x)$ denote the Jacobian of the mapping, i.e.,

$$J(x) = \sqrt{\det(Du(x)(Du(x))^T)}.$$

Then the density of $(u_1(X), \ldots, u_n(X))$ can be written as

$$\tilde{f}(v) = \frac{1}{J(u^{-1}(v))} f(u^{-1}(v)),$$

where $f(\cdot)$ is the density of X.

Let S be the support of X. For any $A \subset S$,

$$\int_A f(x)\,dx = \int_{u(A)} \tilde{f}(v)\,dv,$$

where $u(A)$ denotes the image of set A under the mapping u.

A Change-of-Variable Viewpoint of CMC

Let

$$(u_1(X), \ldots, u_n(X)) = (Y, Z).$$

Denote the density of (Y, Z) by $\tilde{f}(y, z)$.

Problem: to estimate $\alpha = \mathrm{E}[g(X)]$.

Objective: to find a function $r(\cdot)$, such that

$$\alpha = \mathrm{E}[g(X)] = \mathrm{E}[r(Y)].$$

In certain sense, it is equivalent to using CMC by conditioning on Y.

Mathematical Derivation

Note that

$$\mathrm{E}[g(X)] = \int g(x)f(x)\,dx = \int\int g(u^{-1}(y,z))\tilde{f}(y,z)\,dz\,dy.$$

Then we need to choose an appropriate function r such that

$$\alpha = \int\int g(u^{-1}(y,z))\tilde{f}(y,z)\,dz\,dy = \int\int r(y)\tilde{f}(y,z)\,dz\,dy, \tag{1}$$

A Change-of-Variable Representation

It can be verified that one choice of $r(\cdot)$ is

$$r(y) = \frac{\int g(u^{-1}(y,z))\tilde{f}(y,z)\,dz}{\int \tilde{f}(y,z)\,dz}.$$

Under this choice, a new representation of α:

$$\alpha = \mathrm{E}\left[r(Y)\right].$$

Theorem

The new representation always leads to a variance reduction, i.e.,

$$\mathrm{Var}(r(Y)) \leq \mathrm{Var}(g(X)).$$

An Important Special Case

Suppose one wants to estimate

$$\alpha = \mathrm{E}\left[g(X) \cdot 1_{\{h(X) \le z_0\}}\right].$$

We construct a 1-1 mapping as follows:

$$u : (x_1, \ldots, x_m) \to (y_1, \ldots, y_m, z) \equiv \left(\frac{x_1}{h(x)}, \ldots, \frac{x_m}{h(x)}, h(x) \right).$$

Let h_i denote the partial derivative of h w.r.t. x_i. It can be shown that Jacobian of the mapping is

$$J(x) = \sqrt{\frac{1}{h^{2m}} \left(1 - \frac{x_1 h_1 + \ldots + x_m h_m}{h} \right)^2 + \frac{h_1^2 + \ldots + h_m^2}{h^{2(m-1)}}}.$$

Positive Homogeneous h

Suppose h satisfies the following condition:

$$x_1 h_1(x) + \ldots + x_m h_m(x) = h(x). \tag{2}$$

Condition (2) implies positive homogeneity, i.e.,

$$h(tx) = th(x), \quad \forall t \in \mathbb{R}_+.$$

Examples of positive homogeneous functions

- linear functions
- $\max(x_1, \ldots, x_m)$, $\min(x_1, \ldots, x_m)$
- $h(x) = |x|$

Positive Homogeneous h (*cont.*)

When h satisfies Condition (2), the Jacobian has a simpler form:

$$J(x) = \frac{1}{|h^{m-1}(x)|}\sqrt{h_1^2(x) + \ldots + h_m^2(x)}.$$

In this case, derivation of the function $r(\cdot)$ may become much easier.

Many practical problems, e.g., in financial applications, can fit into our framework by using positive homogeneous function h.

Conditional Expectations

In some applications, one may be interested in estimating

$$f_h(z_0), \quad \mathrm{E}[g(X)|h(X) = z_0],$$

where $f_h(\cdot)$ denotes the density of $h(X)$.

Under mild conditions, they can be rewritten as

$$f_h(z_0) = \frac{d}{dz_0}\mathrm{E}\left[1_{\{h(X)\le z_0\}}\right],$$

and

$$\mathrm{E}\left[g(X)|h(X) = z_0\right] = \frac{1}{f_h(z_0)}\frac{d}{dz_0}\mathrm{E}\left[g(X)1_{\{h(X)\le z_0\}}\right].$$

Simulating Conditional Expectations

The problem of simulating conditional expectations then reduces to estimating the following quantity:

$$\beta = \frac{d}{dz_0} \mathrm{E}\left[g(X) 1_{\{h(X) \leq z_0\}} \right].$$

The change-of-variable approach can be applied to establish a new representation of β.

We make use of the mapping

$$u : (x_1, \ldots, x_m) \to (y_1, \ldots, y_m, z) \equiv \left(\frac{x_1}{h(x)}, \ldots, \frac{x_m}{h(x)}, h(x) \right).$$

A New Representation of β

Previous analysis leads to

$$\mathrm{E}\left[g(X)1_{\{h(X)\leq z_0\}}\right] = \mathrm{E}\left[r(Y)\right],$$

where

$$\begin{aligned} r(y) &= \frac{\int g(u^{-1}(y,z))1_{\{z\leq z_0\}}\tilde{f}(y,z)\,dz}{\int \tilde{f}(y,z)\,dz} \\ &= \frac{\int_{\infty}^{z_0} g(u^{-1}(y,z))\tilde{f}(y,z)\,dz}{\int_{-\infty}^{\infty} \tilde{f}(y,z)\,dz}. \end{aligned}$$

A New Representation of β(*cont.*)

Then under appropriate smoothness conditions, we have

$$\beta = \frac{d}{dz_0}\mathrm{E}\left[g(X)1_{\{h(X)\le z_0\}}\right]$$

$$= \frac{d}{dz_0}\mathrm{E}\left[r(Y)\right] = \mathrm{E}\left[\frac{d}{dz_0}r(Y)\right] = \mathrm{E}\left[w(Y)\right],$$

where

$$w(y) = \frac{g(u^{-1}(y,z_0))\tilde{f}(y,z_0)}{\int_{-\infty}^{\infty}\tilde{f}(y,z)\,dz}.$$

Then β can be easily estimated by using samples of $w(Y)$.

Implementation Issues

To derive $r(y)$, one needs the closed-form formulas of $\int_{\infty}^{z_0} g(u^{-1}(y,z))\tilde{f}(y,z)\,dz$ and $\int_{-\infty}^{\infty} \tilde{f}(y,z)\,dz$.

To derive $w(y)$, one needs the closed-form formula of $\int_{-\infty}^{\infty} \tilde{f}(y,z)\,dz$.

This derivation is problem dependent.

It turns out for many practical applications, closed-form formulas can be derived.

An Example

- Let $X = (X_1, \ldots, X_m)$ be underlying asset price valued at m discretization time points.

$$X_k = X_{k-1} \exp\{(r - \sigma^2/2)\tau + \sigma\sqrt{\tau}N_k\}, \quad k = 1, \ldots, m,$$

where $\{N_1, \ldots, N_m\}$ are independent standard normal random variables.

- Let $h(x) = \max(x_1, \ldots, x_m) \equiv \hat{x}$, and $Y = X/h(X)$. The quantities of interest are

$$\alpha = \mathrm{E}\left[1_{\{\hat{X} \le z_0\}}\right] = \mathrm{E}[r(Y)], \quad \beta = \frac{d}{dz_0} E\left[1_{\{\hat{X} \le z_0\}}\right] = \mathrm{E}[w(Y)].$$

An Example (*cont.*)

- It can be derived that

$$r(y) = \Phi\left(\frac{\ln z_0 y_1/x_0 - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right),$$

and

$$w(y) = \frac{1}{z_0\sigma\sqrt{\tau}}\phi\left(\frac{\ln z_0 y_1/x_0 - (r - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right),$$

where Φ and ϕ denote standard normal cdf and pdf resp.

Application to Greek Estimation

- Payoffs of some financial options may be discontinuous. They can usually be written in the form

$$g(X)\prod_{i=1}^{q} 1_{\{l_i(X)\leq a_i\}}.$$

- Here X is a random vector that represents the dynamics of the underlying asset and depends on a market parameter θ.

Expressions of Greeks

The Greek that one wants to estimate can be written as

$$
\begin{aligned}
p'(\theta) = \frac{d}{d\theta}\mathrm{E}\left[g(X)\prod_{i=1}^{q} 1_{\{l_i(X)\le a_i\}}\right] \\
= \mathrm{E}\left[\partial_\theta g(X)\prod_{i=1}^{q} 1_{\{l_i(X)\le a_i\}}\right] \\
-\sum_{i=1}^{q} f_{l_i}(a_i)\mathrm{E}\left[g(X)\partial_\theta l_i(\mathbf{X})\prod_{k\neq i} 1_{\{l_k(X)\le a_k\}}\middle| l_i(X)=a_i\right],
\end{aligned}
$$

where the notation ∂_θ denotes the operator of taking pathwise derivative w.r.t. θ, and $f_{l_i}(\cdot)$ denotes the density of $l_i(X)$.

Expressions of Greeks (*cont.*)

For notational ease, we let

$$g_i(X) = (X)\partial_\theta l_i(X) \prod_{k \neq i} 1_{\{l_k(X) \leq a_k\}}.$$

Then the main problem in Greek estimation reduces to how to estimate

$$\gamma = \sum_{i=1}^{q} f_{l_i}(a_i)\mathrm{E}\left[g_i(X)|\, l_i(X) = a_i\right] = \sum_{i=1}^{q} \frac{d}{da_i}\mathrm{E}\left[g_i(X)1_{\{l_i(X) \leq a_i\}}\right].$$

For most financial options with discontinuous payoffs, their payoffs can be written in the above form using some positive homogeneous functions $l_1, \ldots, l_q$.

Numerical Examples: A Digital Option

- Price dynamics follows a geometric Brownian motion:

$$X_k = X_{k-1} \exp\{(r - \sigma^2/2)\tau + \sigma\sqrt{\tau} N_k\}, \quad k = 1, \dots, m.$$

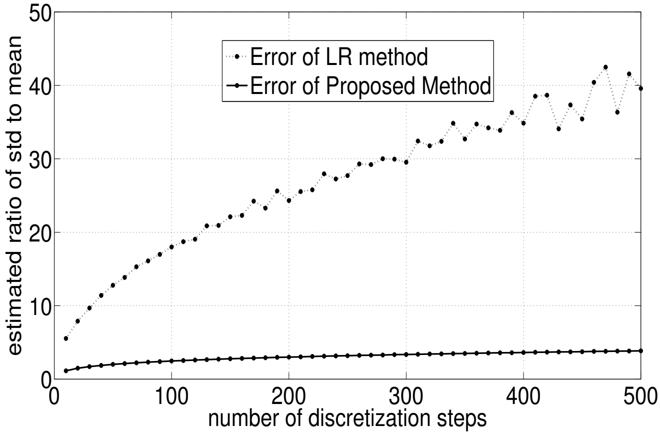
- Discounted payoff: $e^{-rT} 1_{\{X_m \geq K\}}$.

- One wants to estimate *delta*:

$$\Delta = \frac{d}{dx_0} \mathrm{E}\left[e^{-rT} 1_{\{X_m \geq K\}}\right].$$

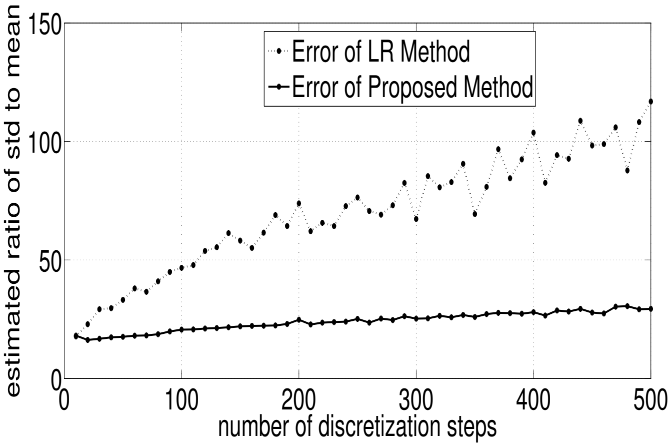
Numerical Results: A Digital Option

We compare the likelihood ratio (LR) method and the change-of-variable method.



Numerical Examples: A Barrier Option

Discounted payoff: $e^{-rT}(X_m - K)^+ 1_{\{\max(X_1,\ldots,X_m)\leq\kappa\}}$.



Conclusions

We have proposed a change-of-variable approach to simulating conditional expectations.

The proposed approach provides a new perspective for understanding conditional Monte Carlo.

It can be used to derive new representations of densities and conditional expectations, which lead to efficient estimators.

As an application, the proposed approach works well in Greek estimation of financial options.