NAME →

ISyE 3770 Solutions — Test 2a — Fall 2007

This test is 80 minutes long. You are allowed one cheat sheet. Do not look at or start the test until you are told to do so. When we ask you to return the test, stop immediately, hand the test in, and do not utter a word to anyone. Do not show any work other than your answers on this sheet. Good luck!

Put your nice, simple answers here...

1. ________ 2. ________ 3. ________

4. ________ 5. ________ 6. ________

7. ________ 8. ________ 9. ________

10. ________ 11. ________ 12. ________

13. ________ 14. ________ 15. ________

16. ________ 17. ________ 18. ________

19. ________ 20. ________ 21. ________

22. ________ 23. ________ 24. ________

25. ________

1. What does the "m" in "p.m.f." mean?

   Solution: mass. $\diamondsuit$

2. Suppose X has p.d.f. $f(x) = x^2$, $0 \le x \le c$. Find c.

   Solution: $\int_0^c f(x)\,dx = 1$ implies $c = 3^{1/3}$. $\diamondsuit$

3. Suppose that X is discrete with $f(-2) = 0.2$, $f(0) = 0.5$, and $f(4) = 0.3$. Find $F(1.5)$, where $F(x)$ is the c.d.f. of X.

   Solution: $F(1.5) = \mathsf{P}(X \le 1.5) = f(-2) + f(0) = 0.7$. $\diamondsuit$

4. Suppose X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. Find $\mathsf{P}(3/4 \le X \le 5/4)$.

   Solution: $\mathsf{P}(3/4 \le X \le 5/4) = \mathsf{P}(3/4 \le X \le 1) = \int_{3/4}^1 2x\,dx = 7/16$. $\diamondsuit$

5. Suppose X has p.d.f. $f(x) = 2x$, $0 \le x \le 1$. Find $\mathsf{P}(0 \le X \le 1/2 \,|\, 1/4 \le X \le 3/4)$.

   Solution:

$$\begin{aligned}
\mathsf{P}(0 \le X \le 1/2 \,|\, 1/4 \le X \le 3/4) &= \frac{\mathsf{P}(0 \le X \le 1/2 \,\cap\, 1/4 \le X \le 3/4)}{\mathsf{P}(1/4 \le X \le 3/4)} \\
&= \frac{\mathsf{P}(1/4 \le X \le 1/2)}{\mathsf{P}(1/4 \le X \le 3/4)} \\
&= \frac{\int_{1/4}^{1/2} f(x)\,dx}{\int_{1/4}^{3/4} f(x)\,dx} \\
&= 3/8. \quad \diamondsuit
\end{aligned}$$

6. If $f(y) = 3y^2$, $0 < y < 1$, find $\mathsf{E}[Y]$.

   Solution: $\int_0^1 y f(y)\,dy = 3/4$. $\diamondsuit$

7. If $f(y) = 3y^2$, $0 < y < 1$, find $\mathsf{E}[1/Y^2]$.

   Solution: $\int_0^1 (1/y^2) f(y)\, dy = 3$. $\diamondsuit$

8. If $\mathsf{E}[X] = 3$ and $\mathsf{E}[X^2] = 10$, find $\mathsf{Var}(X)$.

   Solution: $\mathsf{E}[X^2] - (\mathsf{E}[X])^2 = 1$. $\diamondsuit$

9. True or False? $\mathsf{Var}(X) \geq 0$ for any random variable X.

   Solution: Let $\mu = \mathsf{E}[X]$. Then, by definition, $\mathsf{Var}(X) = \mathsf{E}[(X - \mu)^2]$. This is the expected value of something that cannot be negative. Therefore, the answer is True. $\diamondsuit$

10. If $\mathsf{E}[X] = 3$ and $\mathsf{Var}(X) = 7$, find $\mathsf{E}[2X - 3]$.

    Solution: $\mathsf{E}[2X - 3] = 2\mathsf{E}[X] - 3 = 3$. $\diamondsuit$

11. If $\mathsf{E}[X] = 3$ and $\mathsf{Var}(X) = 7$, find $\mathsf{Var}(2X - 3)$.

    Solution: $\mathsf{Var}(2X - 3) = 4\mathsf{Var}(X) = 28$. $\diamondsuit$

12. If $\mathsf{P}(X = 0) = 0.4$ and $\mathsf{P}(X = 1) = 0.6$, name the distribution of X (including any parameter(s)).

    Solution: Bernoulli(0.6). $\diamondsuit$

13. If $\mathsf{P}(X = 0) = 0.4$ and $\mathsf{P}(X = 1) = 0.6$, find $\mathsf{E}[\ell n(X + 1)]$.

    Solution: $\sum_x \ell n(x + 1) f(x) = \ell n(1) f(0) + \ell n(2) f(1) = \ell n(2) \cdot 0.6 = 0.4159$. $\diamondsuit$

14. What result says that $\mathsf{P}(|X - \mathsf{E}[X]| \geq 1) \leq \mathsf{Var}(X)$? for any random variable X?

    Solution: Chebychev's inequality (with $\sigma^2 = 1$). $\diamondsuit$

15. Suppose that the probability that GT wins any football game in 0.7, and that all games are (somehow) independent. What is the probability that GT will win exactly 3 out of its next 5 games?

    Solution: Let X be the number of games won $\sim$ Binomial$(5, 0.7)$. Then $\mathsf{P}(X = 3) = \binom{5}{3}(0.7)^3(0.3)^2 = 0.3087$. $\diamondsuit$

16. Consider a lightbulb whose lifetime is exponential with a mean of 1 year. What's the probability that the bulb will live at least two years before failing?

    Solution: Let X be the lifetime $\sim$ Exp(1). Then $\mathsf{P}(X > 2) = \int_2^\infty e^{-x}\,dx = e^{-2} = 0.1353$. $\diamondsuit$

17. If $\mathsf{P}(X = 1) = 0.4$ and $\mathsf{P}(X = 3) = 0.6$, find the moment generating function of X, i.e., $\mathsf{E}[e^{tX}]$.

    Solution: By the Law of the Unconscious Statistician, $\mathsf{E}[e^{tX}] = \sum_x e^{tx} f(x) = 0.4e^t + 0.6e^{3t}$. $\diamondsuit$

18. The Geometric(p) distribution is defined to be the number of Bern(p) trials until a success occurs. If $X \sim$ Geom(p), then $\mathsf{P}(X = k) = (1-p)^{k-1}p$, $k = 1, 2, \ldots$. It can also be shown that $\mathsf{E}[X] = 1/p$ for the Geom(p). Anyhow, let X denote the number of trials it takes until "7" comes up when I sum two dice. Find $\mathsf{E}[X]$.

    Solution: $1/p = 6$. $\diamondsuit$

19. If U is a Unif(0,1) random variable, what is the distribution of $-2\ell\text{n}(1 - U)$?

    Solution: From class notes (on functions of a random variable), the distribution is Exp(1/2). $\diamondsuit$

20. If X is a random variable has c.d.f. $F(x)$, what is the distribution of $F(X)$?

    Solution: Unif(0,1). $\diamondsuit$

21. Suppose that X has p.d.f. $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ for $-\infty < x < \infty$. What is the expected value of X?

    Solution: By symmetry, $\mathsf{E}[X] = 0$. $\diamondsuit$

22. What distribution do I get by using the Excel function `RAND()`?

    Solution: Unif(0,1). $\diamondsuit$

23. The *skewness* of a random variable X is a measure of its symmetry. If X has mean μ and variance 1, then the skewness is defined as the third central moment, i.e., $\mathsf{E}[(X-\mu)^3]$. Calculate the skewness of $X \sim \text{Exp}(1)$. **Hint:** From class notes, we know that for the Exp(1) distribution, $\mathsf{E}[X^k] = k!$, for $k = 1, 2, \ldots$.

    Solution:

$$
\begin{aligned}
\mathsf{E}[(X-\mu)^3] &= \mathsf{E}[X^3] - 3\mu\mathsf{E}[X^2] + 3\mu^2\mathsf{E}[X] - \mu^3 \\
&= \mathsf{E}[X^3] - 3\mu\mathsf{E}[X^2] + 2\mu^3 \\
&= 6 - 6 + 2 \;=\; 2. \quad \diamondsuit
\end{aligned}
$$

24. True or False? If X is a positive, continuous random variable, then $\mathsf{E}[X] = \int_0^\infty \mathsf{P}(X > t)\,dt$.

    Solution: True. This follows since

$$
\begin{aligned}
\int_0^\infty \mathsf{P}(X > t)\,dt &= \int_0^\infty \int_t^\infty f(x)\,dx\,dt \\
&= \int_0^\infty \int_0^x f(x)\,dt\,dx \\
&= \int_0^\infty f(x)x\,dx. \quad \diamondsuit
\end{aligned}
$$

Note: To tell you the truth, I would hardly expect you to do such a proof. But you could've done the following: (i) plug in an easy example (say, $\text{Exp}(\lambda)$, (ii) notice that the result is true for your example, and (iii) make the "reasonable" generalization that result is true.

25. Suppose that $X_1, X_2, \ldots, X_n$ are $\text{Bern}(p)$ trials. What's the distribution of $\sum_{i=1}^n X_i$?

Solution: $\text{Binomial}(n, p)$. $\diamondsuit$