NAME →

ISyE 3770 — Test 1*a* Solutions — Fall 2008

This test is 85 minutes long. You are allowed one cheat sheet.

Put your nice, simple answers here...

1(a). ________ 1(b). ________ 1(c). ________

1(d). ________ 1(e). ________ 1(f). ________

1(g). ________ 1(h). ________ 1(i). ________

1(j). ________ 1(k). ________ 1(l). ________

1(m). ________ 1(n). ________ 1(o). ________

2(a). ________ 2(b). ________ 2(c). ________

2(d). ________ 2(e). ________ 2(f). ________

2(g). ________ 2(h). ________ 2(i). ________

2(j). ________

3(a). ________ 3(b). ________ 3(c). ________

4(a). ________ 4(b). ________ 4(c). ________

1. (45 points) Short-Answer Questions

(a) What is the set of all possible outcomes of an experiment called?

Solution: Sample space. $\diamondsuit$

(b) Suppose $\mathsf{P}(\text{I jog today}) = 0.4$, $\mathsf{P}(\text{I watch TV}) = 0.8$, and $\mathsf{P}(\text{I do both}) = 0.3$. What's the probability that I'll do either or both activities?

Solution: Let J = I jog today, T = I watch TV. Then

$$\begin{aligned} \mathsf{P}(\text{I'll do either or both}) &= \mathsf{P}(J \cup T) \\ &= \mathsf{P}(J) + \mathsf{P}(T) - \mathsf{P}(J \cap T) \\ &= 0.4 + 0.8 - 0.3 \\ &= 0.9. \quad \diamondsuit \end{aligned}$$

(c) If $\mathsf{P}(A) = 0.3$ and $\mathsf{P}(B) = 0.5$, and A and B are disjoint, find $\mathsf{P}(A \cap \bar{B})$.

Solution: Since A and B are disjoint, $A \subseteq \bar{B}$ (as is easily verified by a Venn diagram). Thus $A \cap \bar{B} = A$. Thus, $\mathsf{P}(A \cap \bar{B}) = \mathsf{P}(A) = 0.3$. $\diamondsuit$

(d) If $\mathsf{P}(A) = 0.3$ and $\mathsf{P}(B) = 0.5$, and A and B are disjoint, find $\mathsf{P}(A \cup \bar{B})$.

Solution: As in the previous question, $A \subseteq \bar{B}$. Thus, $A \cup \bar{B} = \bar{B}$, and so $\mathsf{P}(A \cup \bar{B}) = \mathsf{P}(\bar{B}) = 1 - P(B) = 1 - 0.5 = 0.5$. $\diamondsuit$

(e) Suppose that $\mathsf{P}(A) = \mathsf{P}(B) = \mathsf{P}(C) = 0.5$ and that A, B, and C are all independent. Find $\mathsf{P}(A \cap B \cap C)$.

Solution: Since A, B, and C are all independent, the probability of their intersection is just the product of their probabilities. That is, $\mathsf{P}(A \cap B \cap C) = \mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C) = 0.5^3 = 0.125$. $\diamondsuit$

(f) Suppose that $\mathsf{P}(A) = \mathsf{P}(B) = \mathsf{P}(C) = 0.5$ and that A, B, and C are all independent. Find $\mathsf{P}(A \cup B \cup C)$.

Solution: We know that in general for three events,

$$\mathsf{P}(A \cup B \cup C) = \mathsf{P}(A) + \mathsf{P}(B) + \mathsf{P}(C) - \mathsf{P}(A \cap B) - \mathsf{P}(A \cap C) - \mathsf{P}(B \cap C) + \mathsf{P}(A \cap B \cap C).$$

Now since the three events are all independent, the probability of the intersections of two or three of them is just the product of the respective probabilities. Thus, we have

$$\mathsf{P}(A \cup B \cup C) = 0.5 + 0.5 + 0.5 - 0.5^2 + 0.5^2 + 0.5^2 + 0.5^3 = 7/8. \quad \diamondsuit$$

(g) TRUE or FALSE? $\mathsf{P}(A) = 0$ implies that A is the null set.

Solution: FALSE. For example, if you pick a random number between 0 and 1, the probability of a particular outcome is zero. $\diamondsuit$

(h) TRUE or FALSE? If $\mathsf{P}(A \cap B \cap C) = \mathsf{P}(A)\mathsf{P}(B)\mathsf{P}(C)$, then A, B, and C are independent.

Solution: FALSE. It could be the case that $\mathsf{P}(A \cap B) \neq \mathsf{P}(A)\mathsf{P}(B)$. $\diamondsuit$

(i) Calculate $\binom{5}{3}$.

Solution: 10. $\diamondsuit$

(j) TRUE or FALSE? $\binom{500}{200} = \binom{500}{300}$.

Solution: TRUE. In general, $\binom{n}{k} = \binom{n}{n-k}$. $\diamondsuit$

(k) Consider the set of letters $\{a, b, c, d, e, f, g, h, i, j\}$. How many distinct 4-letter "words" can you make from this set of 10 letters?

Solution: There are two possible answers, depending on your interpretation. With repetitions allowed: 10^4. $\diamondsuit$

With repetitions not allowed: $10 \times 9 \times 8 \times 7 = P_{10,4} = 5040$. $\diamondsuit$

(l) TRUE or FALSE? If A and B are independent, then $\mathsf{P}(A|B) = \mathsf{P}(B)$.

Solution: FALSE. In fact, if A and B were independent, then we would have $P(A|B) = P(A)$. $\diamondsuit$

(m) How many ways can you arrange the letters in "ARKANSAS"?

Solution: $\frac{8!}{3!2!} = 3360$. $\diamondsuit$

(n) TRUE or FALSE? $\mathsf{P}(A \cap B \cap C) = \mathsf{P}(A|B \cap C)\mathsf{P}(B|C)\mathsf{P}(C)$.

Solution: TRUE. This follows because

$$\begin{aligned} \mathsf{P}(A|B \cap C)\mathsf{P}(B|C)\mathsf{P}(C) &= \frac{\mathsf{P}(A \cap B \cap C)}{\mathsf{P}(B \cap C)} \times \frac{\mathsf{P}(B \cap C)}{\mathsf{P}(C)} \times \mathsf{P}(C) \\ &= \mathsf{P}(A \cap B \cap C). \quad \diamondsuit \end{aligned}$$

(o) Find P(the sum of three fair dice is at most 4).

Solution: The only favorable events are 111, 112, 121 and 211. The total number of possible outcomes is $6^3 = 216$. Therefore, the desired probability is $\frac{4}{216}$. $\diamondsuit$

2. (30 points) Short-Answer Questions on RV's.

   (a) What does "RV" stand for?

   Solution: random variable. $\diamondsuit$

   (b) What does the "d" in "p.d.f." stand for?

   Solution: density. $\diamondsuit$

   (c) Suppose that X is continuous with p.d.f. $f(x) = cx^2$ for $0 < x < 1$. Find c.

   Solution: Since $f(x)$ is a p.d.f., we have

$$1 \;=\; \int_0^1 f(x)\,dx \;=\; \frac{cx^3}{3}\bigg|_0^1 \;=\; \frac{c}{3},$$

   which implies $c = 3$. $\diamondsuit$

   (d) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{P}(0 \le X \le 0.5)$.

   Solution: We have

$$P(0 \le x \le 0.5) = \int_0^{0.5} 2x\,dx = x^2\Big|_0^{0.5} = 0.25. \quad \diamondsuit$$

   (e) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{E}[X]$.

   Solution: We have

$$\mathsf{E}[X] = \int_0^1 2x^2\,dx = \frac{2x^3}{3}\bigg|_0^1 = \frac{2}{3}. \quad \diamondsuit$$

   (f) If X has p.d.f. $f(x) = 2x$ for $0 < x < 1$, find $\mathsf{E}[X^3]$.

   Solution: By the Law of the Unconscious Statistician, we have

$$\mathsf{E}[X^3] = \int_0^1 x^3 \cdot 2x\,dx = \int_0^1 2x^3\,dx = \frac{2x^4}{5}\bigg|_0^1 = \frac{2}{5}. \quad \diamondsuit$$

(g) Suppose X has p.m.f. $f(x) = e^{-\lambda}\lambda^x/x!$ for $x = 0, 1, 2, \ldots$. Name the distribution of X.

Solution: Poisson(λ). $\diamondsuit$

(h) Suppose that X has an Exponential distribution with parameter $\lambda = 3$. Find $\mathsf{E}[2X + 1]$.

Solution: $\mathsf{E}[2X + 1] = 2\mathsf{E}[X] + 1 = 2 \times \frac{1}{3} + 1 = \frac{5}{3}$. $\diamondsuit$

(i) TRUE or FALSE? If X is continuous, then its c.d.f. is the integral of its p.d.f.

Solution: TRUE. $\diamondsuit$

(j) TRUE or FALSE? If X is continuous and always positive, then $\mathsf{E}[X] = \int_0^\infty \mathsf{P}(X > x)\,dx$.

Solution: TRUE. This follows because

$$\begin{aligned}
\int_0^\infty \mathsf{P}(X > x)\,dx &= \int_0^\infty \int_x^\infty f(t)\,dt\,dx \\
&= \int_0^\infty \int_0^t f(t)\,dx\,dt \\
&= \int_0^\infty t\,f(t)\,dt \\
&= \mathsf{E}[X]. \quad \diamondsuit
\end{aligned}$$

3. (10 points) Toss 10 fair dice and let X denote the number of times a “3” appears.

   (a) Name the distribution of X.

   Solution: Binomial$(10, \frac{1}{6})$. $\diamondsuit$

   (b) Is X continuous or discrete?

   Solution: Discrete. $\diamondsuit$

   (c) What's the probability that “3”’ comes up exactly twice?

   Solution:

$$\mathsf{P}(X = 2) = \binom{10}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^8. \quad \diamondsuit$$

4. (15 points) Still More Short-Answer Questions

   (a) Assuming that all 365 birthdays are created equal, what's the probability that at least two people will have a matching birthday if there are 4 people in the room?

   Solution: $\mathsf{P}(\text{at least two}) = 1 - \mathsf{P}(\text{none})$. Thus, we have

   $$\mathsf{P}(\text{at least two people will have a match}) \quad = \quad 1 - \frac{365 \times 364 \times 363 \times 362}{365^4}. \quad \diamondsuit$$

   (b) Suppose we have written 4 letters, but we have randomly inserted them in the 4 corresponding envelopes. (Oops!) What is the probability that at least one of the letters will be in its proper envelope?

   Solution: $1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}$. (This is the classic envelope problem, as described in class.) $\diamondsuit$

   (c) Choose 6 cards from a standard deck. What's the probability that you will get 3 pairs?

   Solution: The total number of hands is $\binom{52}{6}$.
   There are $\binom{13}{3}$ ways to pick the three ranks of the pairs.
   For each pair, there are $\binom{4}{2}$ ways of picking the suits for the two cards that constitute the pair.

   Thus the desired probability is

   $$\frac{\binom{13}{3}\binom{4}{2}^3}{\binom{52}{6}}. \quad \diamondsuit$$